DIFFERENTIAL FORMS, ODD LAPLACIAN AND ”PFAFFIANS“

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Contents

Differential forms on manifold M

$\Leftrightarrow$

Semidensities (half-densities) on odd symplectic superspace $\Pi T^* M$.

The famous Batalin-Vilkovisky operator—Odd Laplacian rightly viewed stands instead of de Rham differential.

This is underlined by some simple and beautiful facts from linear algebra of vector superspaces.

Differential forms on manifolds

Let M be n-dimensional manifold
(local coordinates $(x^1, \ldots, x^n)$.
Differential form on M

$$\Omega(M) \ni \omega = \omega_{i_1 \ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

To view differential forms on manifolds consider vector bundles associated with this manifold.

Spaces ΠT^*M and ΠTM

TM –space of tangent vectors (local coordinates $(x^i, \dot{x}^j)$)

Spaces $\Pi T^* M$ and ΠTM

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$$\{x^i, p_j\} = \delta^i_j, \{x^i, x^j\} = 0, \{p_i, p_j\} = 0\,.$$

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Change parity of fibres

$TM \to \Pi TM$ with coordinates (x^i, ξ^j), ξ^j are odd

$T^*M \to \Pi T^*M$ with coordinates (x^i, θ_j), θ_j are odd
ΠT^*M is an odd symplectic supermanifold endowed with canonical odd symplectic structure:

$$\{x^i, \theta_j\} = \delta^i_j, \{x^i, x^j\} = 0, \{\theta_i, \theta_j\} = 0\,.$$

$$\underbrace{F(x,\theta)}_{\text{function on } \Pi T^*M} = \underbrace{F(x) + F^i(x)\theta_i + F^{ij}\theta_i\theta_j + \cdots + F^{1\ldots n}\theta_1\ldots\theta_n}_{\text{mulitvector field on } M}$$

$$\underbrace{\omega(x,\xi)}_{\text{function on } \Pi TM} = \underbrace{\omega(x) + \omega_i(x)\xi^i + \omega_{ij}\xi^i\xi^j + \cdots + \omega_{1\ldots n}\xi^1\ldots\xi^n}_{\text{differential form on } M}$$

Space of differential forms $\leftrightarrow$ Space of functions on ΠTM

$$\text{e.g. } \omega(x)_{ik} dx^i \wedge dx^k \mapsto \omega(x,\xi) = \omega(x)_{ik}\xi^i\xi^k$$

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Space of multivector fields $\leftrightarrow$ Space of functions on ΠT^*M

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Differential forms $\leftrightarrow$??? on ΠT^*M

Multivector densities $\leftrightarrow$ differential forms

Let σ be density on M ($\approx \sigma = \sigma(x)dx^1 \wedge dx^2 \cdots \wedge dx^n$ n-form)
Let $\mathbf{F} = F^k(x)\partial_k$ be vector field on M (i.e. function $F(x,\theta) = F^k(x)\theta_k$ on ΠT^*M).

$$\text{Vector density } F \otimes \sigma = F^k(x)\frac{\partial}{\partial x^k}\sigma(x)dx^1 \wedge \cdots \wedge dx^n$$

defines $n-1$ form $\omega = *(F\sigma)$

$$\omega(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{n-1}) = \sigma(\mathbf{F}, \mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{n-1})$$

k-multivector density on M $\leftrightarrow$ $n-k$-form on M

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k-multivector density on M $\leftrightarrow$ $n-k$-form on M

$$\underbrace{\omega(x,\xi)}_{\text{function on } \Pi TM} = \int \underbrace{\sigma(x)F(x,\theta)}_{\text{??? on } \Pi T^*M} e^{\xi^k\theta_k} d\theta_1 d\theta_2 \ldots d\theta_n$$

Multivector densitites and semidensities

$$\begin{cases} x^i = x^i(\tilde{x}^1, \ldots, \tilde{x}^n) \\ \theta_j = \frac{\partial \tilde{x}^m(x^1,\ldots,x^n)}{\partial x^j} \tilde{\theta}_m \end{cases}$$

changing of local coordinates on $\Pi T^* M$

$$\mathrm{Ber}\left(\frac{\partial(x,\theta)}{\partial(\tilde{x},\tilde{\theta})}\right) = \mathrm{Ber}\begin{pmatrix} \frac{\partial x^i}{\partial \tilde{x}^k} & \frac{\partial x^r}{\partial x^k}\frac{\partial^2 \tilde{x}^m}{\partial x^j}\theta_m \\ 0 & \frac{\partial \tilde{x}^m(x)}{\partial x^j} \end{pmatrix} = \left(\det\left(\frac{\partial x(\tilde{x})}{\partial \tilde{x}}\right)\right)^2$$

$$\sigma(x)F(x,\theta)dx^1 \wedge dx^n = \sigma(x(\tilde{x}))\det\left(\frac{\partial x(\tilde{x})}{\partial \tilde{x}}\right) d\tilde{x}^1 \ldots d\tilde{x}^n$$

Multivector density $\sigma(x)F(x,\theta)$ is semidensity (half-density) on $\Pi T^* M$

Differential forms and semidensitites

Semidensity

$$s(x,\theta)\sqrt{\mathscr{D}(x,\theta)} = s\left(x(\tilde{x},\tilde{\theta}),\theta(\tilde{x},\tilde{\theta})\right)\left(\mathrm{Ber}\left(\frac{\partial(x,\theta)}{\partial(x,\theta)}\right)\right)^{\frac{1}{2}}\sqrt{\mathscr{D}(\tilde{x},\tilde{\theta})}$$

Differential form on M=Function on $\Pi TM \leftrightarrow$ Semidensity on ΠT^*M

$$\omega(x,\xi) = \int s(x,\theta)e^{\xi^k\theta_k}d\theta_1 d\theta_2 \dots d\theta_n$$

$$s(x,\theta)\sqrt{\mathscr{D}(x,\theta)} = \int \omega(x,\xi)e^{\xi^k\theta_k}d\xi^1 d\xi^2 \dots d\xi^n$$

Odd canonical transformations of $\Pi T^* M$

$$\{x^i, \theta_j\} = \delta^i_j, \{x^i, x^j\} = 0, \{\theta_i, \theta_j\} = 0$$

$$\begin{cases} x^i = x^i(\tilde{x}^1, \dots, \tilde{x}^n) \\ \theta_j = \frac{\partial \tilde{x}^m(x^1,\dots,x^n)}{\partial x^j}\tilde{\theta}_m \end{cases}$$

odd canonic. transformation corresponding to diffeomorphims of M

$$\begin{cases} x^i = \tilde{x}^i + f^i(\tilde{x}, \tilde{\theta}) & (f^i\big|_{\theta=0} = 0) \\ \theta_j = \tilde{\theta}_j + g^j(\tilde{x}, \tilde{\theta}) & (g^j\big|_{\theta=0} = 0) \end{cases}$$

odd canonic. transformation identical on M

$$\begin{cases} x^i = \tilde{x}^i \\ \theta_j = \tilde{\theta}_j + \Psi_j(\tilde{x}) & (\partial_k \Psi_j - \partial_j \Psi_k = 0) \end{cases}$$

special canon. transformation

An arbitrary odd canonical transformation can be considered as a composition of these transformations. (Kh.2000)

De Rham differential in $\Pi T^* M$

Diff.forms = functions on ΠTM

De Rham differential=linear operator on function on ΠTM:

$$d\omega = \xi^i \frac{\partial \omega(x,\xi)}{\partial x^i}$$

$$\begin{array}{ccc} \text{Function on } \Pi TM & \xrightarrow{\tau} & \text{semidensities on } \Pi T^* M \\ d \downarrow & & \Delta^{\#} \downarrow \\ \text{Function on } \Pi TM & \xrightarrow{\tau} & \text{semidensities on } \Pi T^* M \end{array}$$

$$\Delta^{\#}(\tau(\omega)) = \tau(d(\omega)),$$

$$d = \xi^i \frac{\partial}{\partial x^i}, \text{ exterior differential}$$

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Diffeomorphims of $M \subset$ canonical transformations of $\Pi T^* M$

Example

Let $\omega = adx^1 + bdx^2$ on M^2, i.e. $\omega(x, \xi) = a\xi^1 + b\xi^2$ function on ΠTM. Then semidensity $\mathbf{s} = \tau(\omega)$ on ΠT^*M^2 equals to

$$\left(\int (a\xi^1 + b\xi^2) e^{\xi^1\theta_1 + \xi^2\theta_2} d\xi^1 d\xi^2\right) \sqrt{\mathscr{D}(x,\theta)} =$$

$$(a\theta_2 - b\theta_1)\sqrt{\mathscr{D}(x,\theta)}$$

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$$(a\theta_2 - b\theta_1)\sqrt{\mathscr{D}(x,\theta)}$$

$$d\omega = \left(\frac{\partial b}{\partial x^1} - \frac{\partial a}{\partial x^2}\right) dx^1 \wedge dx^2$$

$$\Delta^{\#}(a\theta_2 - b\theta_1)\sqrt{\mathscr{D}(x,\theta)} = \tau(d\omega) =$$

$$\tau\left(\left(\frac{\partial b}{\partial x^1} - \frac{\partial a}{\partial x^2}\right)\xi^1\xi^2\right) = \left(-\frac{\partial b}{\partial x^1} + \frac{\partial a}{\partial x^2}\right)$$

Canonical odd Laplacian on semidensities

Let E be $(n|n)$-dimensional odd symplectic superspace.
(x^i, θ_k) are Darboux coordinates if
$\{x^i, \theta_j\} = \delta^i_j, \{x^i, x^j\} = 0, \{\theta_i, \theta_j\} = 0$.
Then one can define the following canonical operator on semidenisites

$$\Delta^\# \mathbf{s} = \frac{\partial^2 s(x,\theta)}{\partial x^i \partial \theta_i} \sqrt{\mathscr{D}(x,\theta)},$$

where $\mathbf{s} = s(x,\theta)\sqrt{\mathscr{D}(x,\theta)}$ is an expression of semidensity s in

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Canonical odd Laplacian can be considered as a geometrically rightly viewed expression for Batalin-Vilkovisky operator.

Batalin-Vilkovisky Δ-operator

In 1981 I. Batalin and G. Vilkovisky considered the following second-order operator acting on functions on an odd symplectic superspace:

$$\Delta_0 F(x,\theta) = \frac{\partial^2 F(x,\theta)}{\partial x^a \partial \theta_a},$$

where (x^a, θ_a) are arbitrary Darboux coordinates on the odd symplectic superspace. This second order operator is invariant under arbitrary canonical transformations which preserve volume form $dx^1 \dots dx^n d\theta_1 \dots d\theta_n$

$$\underbrace{\{x^1,\dots,x^n;\theta_1,\dots,\theta_n\}}_{\text{Darboux coordinates}} \to \underbrace{\{\tilde{x}^1,\dots,\tilde{x}^n;\tilde{\theta}_1,\dots,\theta_n\}}_{\text{Darboux coordinates}} \text{ such that}$$

$$\mathrm{Ber}\,\frac{\partial(x,\theta)}{\partial(\tilde{x},\tilde{\theta})} = 1\,.$$

Batalin-Vilkovisky identity

For an arbitrary odd canonical transformation

$$\mathrm{Ber}\,\frac{\partial(x,\theta)}{\partial(\tilde{x},\tilde{\theta})} \neq 1\,.$$

This difference with an even canonical transformation is a reason why second order Laplacian arises.

On the other hand the following identity is obeyed:

$$\Delta_0 \sqrt{\left(\mathrm{Ber}\,\frac{\partial(x',\theta')}{\partial(x,\theta)}\right)} = 0\,.$$

This highly non-trivial identity obtained by Batalin and Vilkovisky is a core part of Δ-operators properties.

Invariant construction for BV Δ-operator

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$$D_F = \{f, x^a\}\frac{\partial}{\partial x^a} + \{f, \theta_a\}\frac{\partial}{\partial \theta_a}$$—Hamiltonian vector field

$$\Delta_\rho = \Delta_0, \text{ if } \rho = \mathscr{D}(x,\theta).$$

(Kh. 1989)

Properties of $\Delta - operator$. BV master-equation

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Batalin-Vilkovisky master-equation for the master action $S=\log\sqrt{\rho}$.

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$$\Delta_\rho^2=0\,.$$

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Odd Laplacians on functions and on densities

$$\left(\Delta^{\#}\right)^2 = 0\,.$$

Let ρ be a density (volume form) on an odd symplectic superspace.
Then for an arbitrary function $F = F(x,\theta)$

$$\Delta^{\#}\left(F\sqrt{\rho}\right) = \left(\Delta_\rho F\right)\sqrt{\rho} + (-1)^{p}(F)F\Delta^{\#}\sqrt{\rho}\,.$$

$$\Delta_\rho^2 F = \left\{\frac{1}{\sqrt{\rho}}\Delta^{\#}\sqrt{\rho}, F\right\}\,.$$

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A scalar $\frac{1}{\sqrt{\rho}}\Delta^{\#}\sqrt{\rho}$ is a scalar curvature of a connection which is compatible with the symplectic structure and the volume form (I. Batalin, K. Bering 2006.)

Batalin-Vilkovisky identity (revisited)

Consider semidensity $\mathbf{s} = 1 \cdot \sqrt{\mathscr{D}(x,\theta)}$. By construction

$$\Delta^{\#}\mathbf{s} = \left(\Delta^{\#}\mathbf{s}\right)\sqrt{\mathscr{D}(x,\theta)} = \left(\frac{\partial^2}{\partial x^i \partial \theta_i} 1\right)\sqrt{\mathscr{D}(x,\theta)} =$$

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In new Darboux coordinates $(\tilde{x}, \tilde{\theta})$

$$\mathbf{s} = 1 \cdot \sqrt{\mathscr{D}(x,\theta)} = \sqrt{\left(\mathrm{Ber}\,\frac{\partial(x,\theta)}{\partial(\tilde{x},\tilde{\theta})}\right)}\sqrt{\mathscr{D}(\tilde{x},\tilde{\theta})}\,,$$

$$\Delta^{\#}\mathbf{s} = 0 = \left(\frac{\partial^2}{\partial \tilde{x}^i \partial \tilde{\theta}_i}\sqrt{\left(\mathrm{Ber}\,\frac{\partial(x,\theta)}{\partial(\tilde{x},\tilde{\theta})}\right)}\right)\sqrt{\mathscr{D}(\tilde{x},\tilde{\theta})}\,.$$

Batalin-Vilkovisky identity: $\displaystyle \frac{\partial^2}{\partial \tilde{x}^i \partial \tilde{\theta}_i}\sqrt{\left(\mathrm{Ber}\,\frac{\partial(x,\theta)}{\partial(\tilde{x},\tilde{\theta})}\right)} = 0\,.$

What is linear algebra reasoning of these phenomena?

Recalling: Pfaffian of matrix

Let K be an antisymmetrical matrix:

$$K^{+} = -K\,.$$

Then

$$\det K = (\mathrm{Pf}(K))^2\,,\ \sqrt{\det K} = \mathrm{Pf}(K)\,,$$

where $\mathrm{Pf}(K)$, Pfaffian of matrix K is a polynomial of entries of matrix K

Examples

If *m is an odd number* then $\mathrm{Pf}(K) = 0$, since $\det K = 0$:

$$\det K^{+} = \det K = (-1)^{m} \det K = -\det K\,.$$

$m = 2$

$$K = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix},$$

$$\det K = a^2, \mathrm{Pf}(K) = \sqrt{\det K} = a$$

Examples ($m = 4$)

$$K = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}, \det K = (af + cd - be)^2$$

$$\mathrm{Pf}(K) = af + cd - be = K_{12}K_{34} + K_{14}K_{23} - K_{13}K_{24}\,.$$

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$$F = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & H_z & -H_y \\ -E_y & -H_z & 0 & H_x \\ -E_z & H_y & -H_x & 0 \end{pmatrix}$$

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$$F \wedge F = \mathrm{Pf}(F) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$$

Odd canonical transformations

$n|n$-dimensional odd symplectic superspace:
$\{x^1, \ldots, x^n; \theta_1, \ldots, \theta_n\}$

$$\omega = dx^a d\theta_a \qquad (*)$$

$$\{f, g\} = \frac{\partial f}{\partial x^a}\frac{\partial g}{\partial \theta_a} + (-1)^{p(f)}\frac{\partial f}{\partial \theta_a}\frac{\partial g}{\partial x^a} \qquad (**)$$

$$\{x^a, \theta_b\} = \delta^a_b \, \{x^a, x^b\} = 0, , \{\theta_a, \theta_b\} = 0,$$

$\{x^1, \ldots, x^n; \theta_1, \ldots, \theta_n\}$ are Darboux coordinates

Odd canonical transformation preserve the form (*)
(the odd Poisson bracket (**))

Linear odd canonical transformation

$$(x,\theta)\to(y,\eta)=(x,\theta)\begin{pmatrix} A & \mathscr{B} \\ \mathscr{C} & D \end{pmatrix},\ \begin{cases} y^a = x^b A^a_b + \theta_b \mathscr{C}^b_a \\ \eta_a = x^b \mathscr{B}_{ba} + \theta_b D^b_a \end{cases}$$

where entries of $n\times n$ matrices A and D are even numbers (even elements of a Grassmann algebra), and entries of $n\times n$ matrices $\mathscr{B}$ and $\mathscr{C}$ are odd numbers (odd elements of a Grassmann algebra) and the following conditions are obeyed:

$$\begin{cases} A^+\mathscr{C}+\mathscr{C}^+A=0 \\ D^+\mathscr{B}=\mathscr{B}^+D \\ A^+D+\mathscr{C}^+\mathscr{B}=1 \end{cases}$$

$n|n\times n|n$ matrix $M=\begin{pmatrix} A & \mathscr{B} \\ \mathscr{C} & D \end{pmatrix}$ is an even matrix.

Group and algebra of linear odd canonical transformations

Supergroup $\Pi Sp(n|n)$ and superalgebra $\pi sp(n|n)$.

$$K = \begin{pmatrix} A & \mathcal{B} \\ \mathcal{C} & D \end{pmatrix} \in \Pi Sp(n|n) \qquad \text{if} \quad \begin{cases} A^+\mathcal{C} + \mathcal{C}^+ A = 0 \\ D^+\mathcal{B} = \mathcal{B}^+ D \\ A^+ D + \mathcal{C}^+\mathcal{B} = 1 \end{cases}$$

$$M = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} \in \pi sp(n|n) \qquad \text{if} \quad \begin{cases} \gamma + \gamma^+ = 0 \\ d^+ = d \\ a^+ + d = 0 \end{cases}$$

$$K = e^M \in \Pi Sp(n|n) \text{ if } M \in \pi sp(n|n)\,.$$

(K, M even $n|n \times n|n$ matrices)

Examples

$$K = \begin{pmatrix} A & \mathscr{B} \\ \mathscr{C} & D \end{pmatrix} : \qquad \begin{cases} A^{+}\mathscr{C} + \mathscr{C}^{+}A = 0 \\ D^{+}\mathscr{B} = \mathscr{B}^{+}D \\ A^{+}D + \mathscr{C}^{+}\mathscr{B} = 1 \end{cases}$$

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$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \qquad A^+D = 1$$

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$$\begin{pmatrix} 1 + \mathscr{B}\mathscr{C} & \mathscr{B} \\ \mathscr{C} & 1 \end{pmatrix} \qquad \mathscr{B}^{+} = \mathscr{B}, \mathscr{C}^{+} = -\mathscr{C}\,.$$

Berezinian of an odd canon.transform

Recall formulae for Berezinian (superdeterminant)

$$\mathrm{Ber}\begin{pmatrix} A & \mathscr{B} \\ \mathscr{C} & D \end{pmatrix} = \frac{\det\left(A - \mathscr{B}D^{-1}\mathscr{C}\right)}{\det D},$$

$$\mathrm{Ber}\, e^{\begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}} = e^{\mathrm{Tr}\begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}} = e^{\mathrm{tr} a - \mathrm{tr} d}$$

In a drastic difference to the even case odd canonical transformations do not preserve a volume form.
Berezinian of an odd canonical transformation in general is not equal to unity. If $M=\begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} \in \pi sp(n|n)$ then for matrix $K=e^M \in \Pi Sp(n|n)$

$$\mathrm{Ber}\, e^M = e^{\mathrm{Tr} M} = e^{\mathrm{tr}\, a - \mathrm{tr}\, d} = e^{2\mathrm{tr}\, a}, \text{since } a^+ + d = 0.$$

Example

$$M=\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix},\ K=e^M=\begin{pmatrix} e^a & 0 \\ 0 & e^d \end{pmatrix}=\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix},$$

$a^+ + d = 0 \Rightarrow A^+ D = 1$

$$\mathrm{Ber}\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \frac{\det A}{\det D} = \frac{\det A}{\det (A^+)^{-1}} = \det A^2$$

Fact from linear algebra

Theorem

Let $K = \begin{pmatrix} A & \mathscr{B} \\ \mathscr{C} & D \end{pmatrix}$, *be a matrix of a linear odd canonical transformation. Then*

$$\mathrm{Ber}\, K = (\det A)^2, \sqrt{\mathrm{Ber}\, A} = \det A\,.$$

Polynomial $\det A$ is a square root of Berezinian of odd canonical transformation K ("pfaffian of K").

$$K = K_1 K_2 = \begin{pmatrix} A_1 & \mathscr{B}_1 \\ \mathscr{C}_1 & D_1 \end{pmatrix} \begin{pmatrix} A_2 & \mathscr{B}_2 \\ \mathscr{C}_2 & D_2 \end{pmatrix} = \begin{pmatrix} A_1 A_2 + \mathscr{B}_1 \mathscr{C}_2 & \dots \\ \dots & \dots \end{pmatrix}$$

$$\mathrm{Ber}\, K = \mathrm{Ber}\, K_1 \mathrm{Ber}\, K_2$$

$$\det(A_1 A_2 + \mathscr{B}_1 \mathscr{C}_2) = \det A_1 \det A_2$$

Proof

$$K = \begin{pmatrix} A & \mathscr{B} \\ \mathscr{C} & D \end{pmatrix}: \qquad \begin{cases} A^+\mathscr{C} + \mathscr{C}^+A = 0 \\ D^+\mathscr{B} = \mathscr{B}^+D \\ A^+D + \mathscr{C}^+\mathscr{B} = 1 \end{cases}$$

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$$\begin{pmatrix} 1 + \mathscr{B}\mathscr{C} & \mathscr{B} \\ \mathscr{C} & 1 \end{pmatrix} \qquad \mathscr{B}^+ = \mathscr{B}, \mathscr{C}^+ = -\mathscr{C} \,.$$

Proof...

$$K = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} A' & \mathscr{B} \\ \mathscr{C} & 1 \end{pmatrix}$$

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Proof...

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$$K = \begin{pmatrix} A & 0 \\ 0 & (A^+)^{-1} \end{pmatrix} \begin{pmatrix} 1 + \mathscr{B}\mathscr{C} & \mathscr{B} \\ \mathscr{C} & 1 \end{pmatrix}$$

One can show that $\det(1 + \mathscr{B}\mathscr{C}) = 1$ since $\mathrm{Tr}^k(\mathscr{B}\mathscr{C}) = 0$

$$\mathrm{Ber}\, K = \mathrm{Ber} \begin{pmatrix} A & 0 \\ 0 & (A^+)^{-1} \end{pmatrix} \mathrm{Ber} \begin{pmatrix} 1 + \mathscr{B}\mathscr{C} & \mathscr{B} \\ \mathscr{C} & 1 \end{pmatrix}$$

$$= \frac{\det A}{\det (A^+)^{-1}} \frac{\det(1 + \mathscr{B}\mathscr{C} - \mathscr{B}\mathscr{C})}{\det 1} = \det A^2 .$$

Batalin-Vilkovisky identity (re-revisited)

Consider transformation from Darboux coordinates (x, θ) to Darboux coordinates $(\tilde{x}, \tilde{\theta})$.

$$K = \frac{\partial(x,\theta)}{\partial(\tilde{x},\tilde{\theta})} = \begin{pmatrix} \frac{\partial x}{\partial \tilde{x}} & \frac{\partial \theta}{\partial \tilde{x}} \\ \frac{\partial x}{\partial \tilde{\theta}} & \frac{\partial \theta}{\partial \tilde{\theta}} \end{pmatrix} \in \Pi Sp(n|n) \,.$$

$$\sqrt{\mathrm{Ber}\, K} = \sqrt{\mathrm{Ber}\, \frac{\partial(x,\theta)}{\partial(\tilde{x},\tilde{\theta})}} = \det \frac{\partial x^i}{\partial \tilde{x}^j} \,.$$

$$\Delta_0 \left(\sqrt{\mathrm{Ber}\, \frac{\partial(x,\theta)}{\partial(\tilde{x},\tilde{\theta})}} \right) =$$

$$\frac{\partial^2}{\partial \tilde{x}^i \partial \tilde{\theta}_i} \left(\sqrt{\mathrm{Ber}\, \frac{\partial(x,\theta)}{\partial(\tilde{x},\tilde{\theta})}} \right) = \frac{\partial^2}{\partial \tilde{x}^i \partial \tilde{\theta}_i} \left(\det \frac{\partial x^i}{\tilde{\partial} x^j} \right) = 0 \,.$$

Question: How to describe canonical $\Delta^{\#}$ operator in invariant way?

(The original formula $\Delta^{\#}\mathbf{s} = \frac{\partial^2 s(x,\theta)}{\partial x^a \partial \theta_a}\sqrt{dx^1 \dots dx^p d\theta_1 \dots d\theta_p}$ is written in Darboux coordinates).

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In 2006 K. Bering wrote the explicit expression for $\Delta^{\#}$ operator in an arbitrary coordinates in terms of components of 2-form defining symplectic structure. He proved by straightforward calculations that this expression defines invariant operator which coincides with $\Delta^{\#}$-operator.

(See K. Bering "A Note on Semidensites in Antisymplectic Geometry".hep-th/0604)

Severa's spectral sequence

In 2005 P.Severa constructed the remarkable spectral sequence which contains as ingridients semidensites and $\Delta^{\#}$-operator. Thus he finds a natural definition of this 'somewhat miracolous operator'. (See P. Severa "On the origin of the BV operator..." (math/050633))

Let M be $n|n$-dimensional manifold with symplectic structure defined by odd non-degenerate closed two form ω.

Let $\Omega(M)$ be a space of all (pseudo)differential forms on M, i.e. functions on ΠTM.

Consider differential $Q = d + \omega$. For any F-function on ΠTM (differential form on E) $QF = dF + \omega F$.
One can see that

$$Q^2 = d^2 = \omega^2 = 0, d\omega + \omega d = 0$$

Spectral sequence $\{E_r, d_r\}$

$$E_{r+1} = H(E_r, d_r)$$

with $E_0 = \Omega(M)$, $d_0 = \omega$.

Theorem

The space $E_1 = H(\Omega(M), \omega)$ can be naturally identified with the space of semidensities on M.

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Elements of cohomology space $E_1 = H(\Omega(M), \omega)$ are represented in Darboux coordinates as classes $s(x, \theta)[dx^1 \dots dx^n]$. Under a change of Darboux coordinates $(x, \theta) \to (\tilde{x}, \tilde{\theta})$

$$[dx^1 \dots dx^n] \to \underbrace{\det\left(\frac{\partial x}{\partial \tilde{x}}\right)}$$

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$$[dx^1 \dots dx^n] \to \underbrace{\det\left(\frac{\partial x}{\partial \tilde{x}}\right)}_{\sqrt{\mathrm{Ber}\,\frac{\partial(x,\theta)}{\partial(\tilde{x},\tilde{\theta})}}} [d\tilde{x}^1 \dots d\tilde{x}^n]$$

Theorem

With identification of E_1 with semidensities the differential d_2 of the Severa's spectral sequence vanishes and differential d_3 coincides with the canonical operator $\Delta^{\#}$.
The spectral sequence degenerates at the term E_3.

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The spectral sequence degenerates at the term E_3.

Remark Odd symplectic manifold is symplectomorphic to ΠT^*N, where N is $(n,0)$-dimensional Lagrangian surface in M. $Q = d + \omega$ is twisted differential:

$$QF = e^{-\Theta} d e^{\Theta} F,$$

where $d\Theta = \omega$, ($\Theta = \theta_a dx^a$), Hence

$$H(Q, \Omega(M)) = H(d, M) = H_{\mathrm{de\,Rham}}(N)$$

A.Schwarz, I.Shapiro Twisted de Rham cohomology, homological definition of integral and "Physics over ring" arXiv;0809.0086 [math.AG]

Thank you