BEREZINIANS, RATIONAL FUNCTIONS AND POLYNOMIAL INVARIANTS

Hovhannes Khudaverdian

University of Manchester, Manchester, UK

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Contents

Even matrices

Let K be an even $p|q \times p|q$ matrix

$$K = \begin{pmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{pmatrix},$$

where entries of $p \times p$ matrix K_{00} and $q \times q$ matrix K_{11} are even numbers (even elements of a Grassmann algebra), and entries of $p \times q$ matrix K_{01} and $q \times p$ matrix K_{10} are odd numbers (odd elements of a Grassmann algebra)

$$\mathrm{Tr}\, K = \mathrm{tr}\, K_{00} - \mathrm{tr}\, K_{11}$$

Invariant polynomials

$F = F(K)$ is an invariant function on entries of $p|q \times p|q$ matrices K if

$$F(C^{-1}KC) = F(K)\,.$$

Example

Polynomials $s_r(K) = \mathrm{Tr}\, K^r$ are invariant polynomials:

$$\mathrm{Tr}\,(C^{-1}KC) = \mathrm{Tr}\,(K)\,.$$

How to describe space of all invariant polynomials (invariant rational functions) on $p|q \times p|q$ even matrices?

Bosonic case (q=0)

Textbook answer: the ring of invariant polynomials is freely generated by polynomials $\{s_1(K), \dots, s_p(K)\}$.

Example

$F(K) = \det K$ on 2×2 matrices is invariant polynomial:

$$\det K = \frac{s_1^2(K) - s_2(K)}{2} = \frac{\mathrm{Tr}^2 K - \mathrm{Tr}\, K^2}{2}.$$

Corollary

The field of invariant rational functions is also generated by polynomials $\{s_1(K), \dots, s_p(K)\}$.

Theorem on ring of invariant polynomials

Proposition
The field of invariant rational functions on even $p|q \times p|q$ matrices K is generated by the finite set of $p+q$ polynomials $\{s_m(K) = \mathrm{Tr}\, K^m\}$ ($m = 1, 2, 3, \ldots, p+q$).

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Theorem
The ring of invariant polynomials on even $p|q \times p|q$ matrices K is generated by the infinite set of polynomials $\{s_m(K) = \mathrm{Tr}\, K^m\}$ ($m = 1, 2, 3, \ldots$) .
This ring is not generated by a finite set of polynomials
(There is an infinite set of relations between $s_m(K)$.)

The Theorem is a not-trivial statement. It states that if $F(K)$ is an invariant polynomial, then

$$F(K) = G(s_1, s_2, s_3, \dots, s_N)\big|_{s_r = s_r(K) = \mathrm{Tr}\, K^r} ,$$

where N could be arbitrary large. E.g. the polynomial $F(K) = \mathrm{Tr}\, K^{N+1}$ cannot be expressed via polynomial of variables $\{s_1, s_2, s_3, \dots, s_N\}$, but it can be expressed as rational function on polynomials $\{s_1, s_2, s_3, \dots, s_{p+q}\}$.

Example: the ring is not finitely generated!

$$K = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}, \; a, d \text{ are even, } \beta, \gamma \text{ are odd}$$

K is $1|1 \times 1|1$ even matrix

$$s_1(K) = \mathrm{Tr}\, K = a - d,\; s_2(K) = \mathrm{Tr}\, K^2 = a^2 - d^2 + 2\beta\gamma, \dots$$

$s_{100}(K) = \mathrm{Tr}\, K^{100}$ cannot be expressed as a polynomial on generators $s_1, s_2, \dots, s_{99}$.

It can be expressed as a rational function on generators s_1, s_2.

If (λ, μ) are eigenvalues of matrix K then

$$\lambda, \mu = \frac{1}{2}\left(\frac{\lambda^2 - \mu^2}{\lambda - \mu} \pm (\lambda - \mu)\right) = \frac{1}{2}\left(\frac{s_2(K)}{s_1(K)} \pm s_1(K)\right), \text{ thus}$$

$$s_{100}(K) = \lambda^{100} - \mu^{100} = \frac{1}{2^{100}}\left(\frac{s_2(K)}{s_1(K)} + s_1(K)\right)^{100} - \left(\frac{s_2(K)}{s_1(K)} - s_1(K)\right)^{1}$$

Reduction to the case of polynomials on eigenvalues

To understand the essence of the statement of Theorem consider the restrictions of invariant functions on $p|q \times p|q$ diagonalisable matrices.
An invariant polynomial F on $p|q \times p|q$ even matrices defines the polynomial G_F on $p+q$ variables $(\lambda_1, \ldots, \lambda_p; \mu_1, \ldots, \mu_q)$

$$G_F(\lambda_1, \ldots, \lambda_p; \mu_1, \ldots, \mu_q) = F\left(\mathrm{diag}\left[\lambda_1, \ldots, \lambda_p; \mu_1, \ldots, \mu_q\right]\right).$$

How the invariance of polynomials $F(K)$ is inherited by polynomials $G_F = G_F(\lambda_1, \ldots, \lambda_p; \mu_1, \ldots, \mu_q)$?

Polynomials G_F are $S_p \times S_q$ invariant

The invariance of polynomial F implies the fact that polynomial $G_F(\lambda_1,\dots,\lambda_p;\mu_1,\dots,\mu_q)$ is symmetric polynomial on variables $\{\lambda_i\}$ and it is symmetric polynomial on variables $\{\mu_\alpha\}$:

$$G_F(\dots,\lambda_i,\dots,\lambda_j\dots;\mu_1\dots,\mu_q) = G_F(\dots,\lambda_j,\dots,\lambda_i\dots;\mu_1\dots,\mu_q)$$

$$G_F(\lambda_1,\dots,\lambda_p;\dots\mu_\alpha,\dots,\mu_\beta,\dots) = G_F(\lambda_1,\dots,\lambda_p;\dots,\mu_\beta,\dots,\mu_\alpha,\dots)$$

(We call this condition $S_p \times S_q$-invariance).
$S_p \times S_q$-invariance is necessary but not sufficient condition.

Berezin-Sergeev (BS) condition

If F is invariant polynomial then polynomial $G_F(\lambda_1, \ldots, \lambda_p; \mu_1, \ldots, \mu_q)$ obeys the following condition:

$$\left(\frac{\partial G_F}{\partial \lambda_i} + \frac{\partial G_F}{\partial \mu_\alpha}\right)\Big|_{\lambda_i = \mu_\alpha} = 0, \text{ i.e.}$$

$$G_F(\lambda_1, \ldots, \lambda_p; \mu_1, \ldots, \mu_q)\Big|_{\lambda_i = \mu_\alpha = t} \quad \text{does not depend on } t$$

for an arbitrary λ_i, μ_a $(i = 1, \ldots, p, \alpha = 1, \ldots, q)$.
We call this condition Berezin—Sergeev (BS) condition.
This is highly not-trivial condition.
Explain where it comes from

Where BS conditions comes from

Let $K = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}$, be an even $1|1 \times 1|1$ matrix, (a, d are even, α, δ are odd). If $a \neq d$ then this matrix is diagonalisable:

$$CKC^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix},$$

$$\lambda = a + \frac{\beta\gamma}{a-d},\ \mu = d + \frac{\beta\gamma}{a-d}.$$

$$F(K) = G_F(\lambda, \mu) = G_F\left(a + \frac{\beta\gamma}{a-d}, d + \frac{\beta\gamma}{a-d}\right)$$

$$= G_F(a,d) + \frac{\alpha\delta}{a-d}\left(\frac{\partial P_F(a,d)}{\partial a} + \frac{\partial P_F(a,d)}{\partial d}\right).$$

BS condition implies that $F(K)$ is a polynomial.

Basis in the space of polynomials

Polynomials

$$s_r(K) = \mathrm{Tr}\, K^r = \lambda_1^r + \lambda_2^r + \cdots + \lambda_p^r - \mu_1^r - \mu_2^r - \cdots - \mu_p^r, \; (r = 0, 1, 2, 3, \dots)$$

are $S_p \times S_q$–invariant and they obey Berezin-Sergeev condition.

Is it true that an arbitrary $S_p \times S_q$-invariant polynomial obeying BS conditions is a polynomial on $\{s_r\}$?

Berezin-Sergeev Theorem

Theorem

(Berezin, Sergeev) The ring of $S_p \times S_q$-invariant polynomials on $p+q$ variables $(\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q)$ which obey the Berezin-Sergeev condition is generated by polynomials

$$s_r = \lambda_1^r + \dots + \lambda_p^r - \mu_1^r - \mu_2^r - \dots - \mu_p^r, (r = 1, 2, 3, \dots,)$$

This theorem is equivalent to the Theorem on invariant polynomials.
F is invariant polynomial on $p|q \times p|q$ matrices $\Leftrightarrow$ G_F is $S_p \times S_q$-invariant polynomial obeying *BS* condition, i.e. $G_F = G_F(s_1, \dots, s_K)$,

$$F(K) = P_F(s_1, \dots, s_K)\big|_{s_r = \mathrm{Tr}\, K^r} .$$

not Noetherian ring

Ring of $S_p \times S_q$-invariant polynomials obeying Berezin-Sergeev condition is not finitely generated ring and it is not Noetherian ring.
Let $p = q = 1$. Then $s_m(\lambda, \mu) = \lambda^m - \mu^m$. The generators $\{s_m\}$ satisfy infinite set of relations. The BS-condition means that $P(\lambda, \mu) = c + (\lambda - \mu)G(\lambda, \mu)$. This ring obviously is not finitely generated and the ideal $J = \{P(\lambda, \mu) \colon \ P(0,0) = 0\}$ is not finitely generated (over the ring).

$c_k = (-1)^{k-1}\mu^{k-1}(\lambda - \mu)$ obey relations $c_{k-1}c_{k+1} = c_k^2$.

$$\left(c_k = \mathrm{Tr} \wedge^k \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \right)$$

Setup

We consider this problem from another point of view.
Let K be a linear operator on $p|q$-dimensional superspace V
Consider characteristic function

$$R_K(z) = \mathrm{Ber}\,(1 + zK)$$

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We consider this problem from another point of view.
Let K be a linear operator on $p|q$-dimensional superspace V
Consider characteristic function

$$R_K(z) = \mathrm{Ber}\,(1 + zK)$$

Characteristic function is rational function

$$R_K(z) = \mathrm{Ber}\,(1 + zK) = \frac{1 + a_1 z + a_2 z^2 + \cdots + a_p z^p}{1 + b_1 z + b_2 z^2 + \cdots + b_q z^q}$$

What is superspace

$V = V_0 \oplus V_1$ Z_2-graded vector space, $\dim V_0 = p$, $\dim V_1 = q$.
Assign to an arbitrary Grassmann algebra $\Lambda = \Lambda_0 \oplus \Lambda_1$ the set

$$V_\Lambda = V_0 \otimes \Lambda_0 \oplus V_1 \otimes \Lambda_1$$

$$V_\Lambda \ni x = \sum_{i=1}^{p} a^i \mathbf{e}_i + \sum_{\alpha=1}^{q} b^\alpha \mathbf{f}_\alpha$$

Superspace V is a functor $\Lambda \to V_\Lambda$.
V_Λ is a set of Λ-points of the superspace V.

Berezinian (superdeterminant)

Let K be linear operator on V. Its matrix: $\begin{pmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{pmatrix}$

Entries of matrices K_{00} and K_{11} are even elements of Λ.
Entries of matrices K_{01} and K_{10} are odd elements of Λ.

$$\mathrm{Ber}\, K = \frac{\det\left(K_{00} - K_{01} K_{11}^{-1} K_{10}\right)}{\det K_{11}}, \qquad \mathrm{Tr}\, K = \mathrm{tr}\, K_{00} - \mathrm{tr}\, K_{11}$$

$$\mathrm{Ber}\,(AB) = \mathrm{Ber}\, A \cdot \mathrm{Ber}\, B. \qquad \mathrm{Ber}\, \exp K = \exp\left(\mathrm{Tr}\, K\right)$$

Characteristic function is rational function

$$R_K(z) = \mathrm{Ber}\,(1+zK) = \frac{1+a_1 z + a_2 z^2 + \cdots + a_p z^p}{1+b_1 z + b_2 z^2 + \cdots + b_q z^q}$$

If $K = \mathrm{diag}[\lambda_1, \ldots \lambda_p; \mu_1, \ldots \mu_q]$ then

$$\mathrm{Ber}\,(1+zK) = \frac{\prod_{i=1}^{p}(1+z\lambda_i)}{\prod_{\alpha=1}^{q}(1+z\mu_\alpha)}$$

$$a_1 = \lambda_1 + \cdots + \lambda_p, \ldots, a_p = \lambda_1\lambda_2\ldots\lambda_p,$$
$$b_1 = \mu_1 + \cdots + \mu_p, \ldots, b_q = \mu_1\mu_2\ldots\mu_q.$$

$$\mathrm{Ber}\,(1+zK) = \frac{1+a_1 z + a_2 z^2 + \cdots + a_p z^p}{1+b_1 z + b_2 z^2 + \cdots + b_q z^q} = 1 + c_1 z + c_2 z^2 + \ldots,$$

$$c_1 = a_1 - b_1 = \sum \lambda_i - \sum \mu_\alpha = \mathrm{Tr}\,K,$$

$$c_2 = a_2 - b_1 c_1 - b_2 = \sum_{i<j} \lambda_i\lambda_j - \sum_{i,\alpha} \lambda_i \mu_\alpha + \sum \mu_\alpha\mu_\beta - \sum_{\alpha<\beta} \mu_\alpha\mu_\beta = \mathrm{Tr}\,(K \wedge K)$$

$$\ldots$$

Polynomials $c_r(K)$ and $s_r(K)$

$$c_r = \mathrm{Tr} \wedge^r K, \quad r = 1, 2, 3, \ldots$$

In a pure bosonic case ($q = 0$) $c_r = 0$ for $r > p$.
In a supercase c_r form a recurrent sequence of the period q.
Polynomials $c_r(K) = \mathrm{Tr} \wedge^r K$ can be expressed via $s_r(K) = \mathrm{Tr}\, K^r$ (and vice versa s_r can be expressed via c_r) by universal recurrent polynomial relations:

$$c_1 = s_1, c_2 = \frac{s_1^2 - s_2}{2}, c_3 = \frac{1}{6}(s_1^3 - 3s_1 s_2 + 2s_3), \ldots$$

$$c_{m+1} = \frac{s_1 c_m - s_2 c_{m-1} + \cdots + (-1)^m s_{m+1}}{m+1}$$

Invariant polynomials in terms of characteristic function

An invariant polynomial $F = F(K)$ defines the polynomial G_F:

$$G_F(\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q) = F\left(\mathrm{diag}\left[\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q\right]\right), .$$

It defines the polynomial L_F on coefficients of numerator and denominator of characteristic function

$$R_K(z) = \mathrm{Ber}\,(1 + zK) = \frac{1 + a_1 z + a_2 z^2 + \dots + a_p z^p}{1 + b_1 z + b_2 z^2 + \dots + b_q z^q} = \frac{P(z)}{Q(z)}$$

$$L_F(P(z), Q(z)) = L_F(a_1, \dots, a_p, b_1, \dots, b_q) = G_F(\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q),$$

$$a_1 = \lambda_1 + \dots + \lambda_p, \dots, a_p = \lambda_1 \cdot \dots \cdot \lambda_p, \dots,$$
$$b_1 = \mu_1 + \dots + \mu_q, \dots, b_q = \mu_1 \cdot \dots \cdot \mu_q.$$

Conditions for polynomial L_F

$F(K)$ is invariant polynomial $\Leftrightarrow$ $S_p \times S_q$-invariant polynomial $G_F(\lambda_1, \ldots, \lambda_p; \mu_1, \ldots, \mu_q)$ obeys BS condition $\Leftrightarrow$ the polynomial $L_F = L_F(P(z), Q(z))$ depends on ratio of two polynomials, i.e., on rational function:

$$L_F(P(z), Q(z)) = L_F\left(\frac{P(z)}{Q(z)}\right)$$

$$L_F(P_1(z), Q_1(z)) = L_F(P_2(z), Q_2(z)) \qquad \text{if} \qquad \frac{P_1(z)}{Q_1(z)} = \frac{P_2(z)}{Q_2(z)}$$

This is the condition which stands instead Berezin–Sergeev condition:

$$\begin{matrix} P_t(z) = P_0(z)(1 + tz) \\ Q_t(z) = Q_0(z)(1 + tz) \end{matrix}, \Rightarrow \frac{P_t(z)}{Q_t(z)} = \frac{P_{t'}(z)}{Q_{t'}(z)} = \frac{P_0(z)}{Q_0(z)}$$

Useful notations

- A space of pairs of (normalised) polynomials

$$\mathscr{A} = \{(P, Q),\ P(z)\big|_{z=0} = Q(z)\big|_{z=0} = 1\}$$

- A space of pairs of (normalised) polynomials of degree p, q

$$\mathscr{A}_{p,q} = \{(P, Q),\ P = 1 + a_1 z + \cdots + a_p z^p\ , Q = 1 + b_1 z + \cdots + b_q z^q\}$$

- The space A of (normalised) fractions $\frac{P(z)}{Q(z)}$, $(P, Q) \in \mathscr{A}$

$$A = \mathscr{A}/_{\approx}:\ \ ((P, Q) \approx (P', Q') \text{ if } PQ' = P'Q \text{ i.e. } \frac{P(z)}{Q(z)} = \frac{P'(z)}{Q'(z)}$$

- The space $A_{p|q}$ of (normalised) fractions $\frac{P(z)}{Q(z)}$, $(P, Q) \in \mathscr{A}_{p,q}$,

$$A_{p|q} = \mathscr{A}_{p,q}/_{\approx}:\ \ (P, Q) \approx (P', Q') \text{ if } PQ' = P'Q \text{ i.e. } \frac{P(z)}{Q(z)} = \frac{P'(z)}{Q'(z)}$$

Conditions on $L_F(P,Q) \Leftrightarrow$ Invariance of $F(K)$

- $F(K)$ is inv. polynomial on $p|q \times p|q$ even matrices
- $G_F(\lambda_1,\dots,\lambda_p;\mu_1,\dots,\mu_q)$ is $S_p \times S_q$-inv. polynomial and

  $G_F\big|_{\lambda_i=\mu_\alpha=t}$ does not depend on t (Berezin-Sergeev condition)

- Pol-al $L_F(P(z),Q(z))$ on $\mathscr{A}_{p,q}$ is well-defined on $A_{p|q}$.

$$L_F(P(z),Q(z)) = L_F\left(\frac{P(z)}{Q(z)}\right)$$

These conditions are equivalent.

We say polynomial $L(P(z),Q(z))$ is invariant on $\mathscr{A}_{p,q}$ if it is well-defined on $A_{p|q}$.

Example of invariant polynomial on $\mathscr{A}_{2,2}$. I

$$F(K) = \mathrm{Tr}\, K^2 \text{ on } 2|2 \times 2|2 \text{ even matrices.}$$

$$G_F(\lambda_1, \lambda_2, \mu_1, \mu_2) = \lambda_1^2 + \lambda_2^2 - \mu_1^2 - \mu_2^2 \,.$$

$$\mathrm{Ber}\,(1 + Kz) = \frac{(1+\lambda_1 z)(1+\lambda_2 z)}{(1+\mu_1 z)(1+\mu_2 z)} = \frac{1 + a_1 z + a_2 z^2}{1 + b_1 z + b_2 z^2}, = 1 + c_1 z + c_2 z^2 + .$$

$$L_F(P(z), Q(z)) = (a_1^2 - 2a_2) - (b_1^2 - 2b_2) \text{ polynomial on } \mathscr{A}_{2,2}$$

$$L_F(P, Q) = \left(\underbrace{(1+dz)(1+tz)}_{P = 1 + (d+t)z + dtz^2}, \underbrace{(1+fz)(1+tz)}_{Q = 1 + (f+t)z + ftz^2} \right) =$$

$$(d+t)^2 - 2dt - (f+t)^2 + 2ft = d^2 - f^2 = L_F\left(\frac{P}{Q}\right)$$

L_F is an invariant polynomial on $\mathscr{A}_{2,2}$, i.e. it is a polynomial well-defined on $A_{2|2}$.

Theorem on ring of invariant polynomials

Equivalent formulations

- Invariant polynomials $F(K)$ depend on $s_r = \mathrm{Tr}\, K^r$, or on $c_r = \mathrm{Tr} \wedge^r K$
- $S_p \times S_q$ invariant-polynomial $G(\lambda_i, \mu_\alpha)$ obeying Berezin–Sergeev condition depends on $s_r = \sum_i \lambda_i^r - \sum_i \mu_\alpha^r$.

Theorem on ring of invariant polynomials

Equivalent formulations

- Invariant polynomials $F(K)$ depend on $s_r = \mathrm{Tr}\, K^r$, or on $c_r = \mathrm{Tr} \wedge^r K$
- $S_p \times S_q$ invariant-polynomial $G(\lambda_i, \mu_\alpha)$ obeying Berezin–Sergeev condition depends on $s_r = \sum_i \lambda_i^r - \sum_i \mu_\alpha^r$.
- An invariant polynomial $L(P, Q)$ on $\mathscr{A}_{p,q}$ (a well-defined function on $A_{p|q}$) is a restriction of an invariant polynomial on $\mathscr{A}$ (a well-defined function on A)

Polynomials on rational functions.

A polynomial $L^*(c_1, c_2, \ldots, c_N)$ defines polynomial function on a space A of (normalised) fractions:

$$\frac{P(z)}{Q(z)} = \frac{1 + a_1 z + a_2 z^2 + \cdots + a_m z^m}{1 + b_1 z + b_2 z^2 + \cdots + b_n z^n} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots$$

$$\text{then } L\left(\frac{P}{Q}\right) = L^*(c_1, c_2, \ldots, c_N)\Big|_{c_i = c_i\left(\frac{P}{Q}\right)}, \qquad (*)$$

$$c_1 = a_1 - b_1, c_2 = a_2 - c_1 b_1 - b_2, c_3 = a_3 - c_1 b_2 - c_2 b_1 - b_3, \ldots$$

Example. $L^* = c_1^2 - c_2$.

$L\left(\frac{1+az}{1+b_1 z+b_2 z^2}\right) = (a - b_1)^2 - (b_1^2 - ab_1 - b_2)$ since

$$\frac{1 + az}{1 + b_1 z + b_2 z^2} = 1 + (a - b_1)z + (b_1^2 - ab_1 - b_2)z^2 + \ldots$$

$L^*(c_1, c_2, \ldots, c_N)$ is inv. polynomial on $\mathscr{A}$. Its restriction (*) defines inv. polynomial on $\mathscr{A}_{p,q}$, i.e. a function on $A_{p|q}$.

Formulation of Theorem in terms of functions on fractions

Theorem

Any polynomial on $\mathscr{A}_{p,q}$ such that it is well-defined on $A_{p|q}$ is a restriction of a polynomial on A.

$$L(P,Q)\colon \forall (P',Q') \in \mathscr{A}_{p-1,q-1}$$

$$L(P'(1+tz), Q'(1+tz)) \textit{ does not depend on } t \left(L = L\left(\frac{P}{Q}\right)\right)$$

$$\exists\, L^* = L^*(c_1,\dots,c_N), \text{ such that } L(P,Q) = L^*(c_1,c_2,\dots,c_N)\big|_{c_i=c_i\left(\frac{P}{Q}\right)}$$

$$\frac{P(z)}{Q(z)} = \frac{1+a_1z+a_2z^2+\dots+a_pz^p}{1+b_1z+b_2z^2+\dots+b_qz^q} = 1+c_1z+c_2z^2+c_3z^3+\dots$$

This is an equivalent reformulation

Let $F(K)$ be invariant polynomial on $p|q \times p|q$ even matrices.
Consider a polynomial $L(P, Q)$ on $\mathscr{A}_{p,q}$ defined by

$$L(P, Q) = F(K) \text{where } \frac{P(z)}{Q(z)} = \mathrm{Ber}\,(1 + zK),\ L(P, Q) = L_F(P, Q).$$

$$L(P, Q) = L^*(c_1, c_2, \dots, c_N)\big|_{c_i = c_i\left(\frac{P}{Q}\right)} \Leftrightarrow F(K) = L^*(c_1, c_2, \dots, c_N)\big|_{c_i = \mathrm{Tr} \wedge^i K}$$

Example of invariant polynomial on $\mathscr{A}_{2,2}$.II

$$F(K) = \operatorname{Tr} K^2 \text{ on } 2|2 \times 2|2 \text{ even matrices.}$$

$$G_F(\lambda_1, \lambda_2, \mu_1, \mu_2) = \lambda_1^2 + \lambda_2^2 - \mu_1^2 - \mu_2^2 .$$

$$\operatorname{Ber}(1 + Kz) = \frac{(1+\lambda_1 z)(1+\lambda_2 z)}{(1+\mu_1 z)(1+\mu_2 z)} = \frac{1 + a_1 z + a_2 z^2}{1 + b_1 z + b_2 z^2}, = 1 + c_1 z + c_2 z^2 + .$$

$L_F(P(z), Q(z)) = (a_1^2 - 2a_2) - (b_1^2 - 2b_2)$ is well defined on $A_{2,2}$

Consider $L^* = c_1^2 - 2c_2$. $c_1 = a_1 - b_1$, $c_2 = a_2 - c_1 b_1 - b_2$

$$L^*\big|_{c_i = c_i(\frac{P}{Q})} = (a_1 - b_1)^2 - 2(a_2 - (a_1 - b_1)b_1 - b_2) =$$

$$a_1^2 - 2a_2 - b_1^2 + 2b_2 = L_F(P, Q)$$

Example of invariant polynomial on $\mathscr{A}_{2,2}$.II

$$F(K) = \mathrm{Tr}\, K^2 \text{ on } 2|2 \times 2|2 \text{ even matrices.}$$

$$G_F(\lambda_1, \lambda_2, \mu_1, \mu_2) = \lambda_1^2 + \lambda_2^2 - \mu_1^2 - \mu_2^2 .$$

$$\mathrm{Ber}\,(1 + Kz) = \frac{(1+\lambda_1 z)(1+\lambda_2 z)}{(1+\mu_1 z)(1+\mu_2 z)} = \frac{1 + a_1 z + a_2 z^2}{1 + b_1 z + b_2 z^2}, = 1 + c_1 z + c_2 z^2 + .$$

$L_F(P(z), Q(z)) = (a_1^2 - 2a_2) - (b_1^2 - 2b_2)$ is well defined on $A_{2,2}$

Consider $L^* = c_1^2 - 2c_2$. $c_1 = a_1 - b_1$, $c_2 = a_2 - c_1 b_1 - b_2$

$$L^*\big|_{c_i = c_i(\frac{P}{Q})} = (a_1 - b_1)^2 - 2(a_2 - (a_1 - b_1)b_1 - b_2) =$$

$$a_1^2 - 2a_2 - b_1^2 + 2b_2 = L_F(P, Q)$$

$$(c_1^2 - 2c_2)\big|_{c_i = \mathrm{Tr}\wedge^i K} = \mathrm{Tr}^2 K - 2\mathrm{Tr} \wedge^2 K = \mathrm{Tr}\, K^2$$

Weight of polynomials

$$\sigma(a_k) = \sigma(b_k) = \sigma(c_k) = k$$

If $(P, Q) \in \mathscr{A}_{r,s}$ then $L = L(P, Q) = L(a_1, a_2, \ldots, a_r; b_1, \ldots, b_s)$

$$(P, Q) = (1 + a_1 z + a_2 z^2 + \cdots + a_r z^r, 1 + b_1 z + b_2 z^2 + \cdots + b_s z^s)$$

$$L(P, Q) = a_1^2 - 2a_2 - b_1^2 + 2b_2, \ \sigma(L) = 2.$$

Key lemma

The theorem follows from the following

Lemma

Let $L(P,Q)$ be an invariant polynomial on $\mathscr{A}_{r,s}$, i.e. it is well-defined on $A_{r|s}$. Then there exists an invariant polynomial L' on $\mathscr{A}_{r+1,s+1}$, i.e. the polynomial which is well-defined on $A_{r+1|s+1}$ such that $\sigma(L') = \sigma(L)$ and L is the restriction of L' on $\mathscr{A}_{r,s}$:

$$(P,Q) \in \mathscr{A}_{r,s},\ L(P,Q) = L'(P(1+tz), Q(1+tz)) \text{ i.e.}$$

$$\frac{P}{Q} \in A_{r|s},\ L\left(\frac{P}{Q}\right) = L'\left(\frac{P}{Q}\right).$$

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$$L' \to L' + \mathrm{Res}\,(P,Q)F(P,Q)$$

Proof of the Theorem

Let $L = L(P, Q)$ be an invariant polynomial on $\mathscr{A}_{3,2}$, i.e. $L = L\left(\frac{P}{Q}\right)$ on $A_{3|2}$ and $\sigma(L) = 15$.

Using lemma consider prolongation of L on polynomial $\tilde{L}$ on $A_{103|102}$

$$A_{3|2} \longrightarrow A_{4|3} \longrightarrow A_{5|4} \longrightarrow \dots \longrightarrow A_{102|101} \longrightarrow A_{103|102}$$

$$\tilde{L} = \tilde{L}\left(\frac{P}{Q}\right) = \tilde{L}(a_1, a_2, \dots, a_{103}; b_1, \dots, b_{102})$$

on $A_{103|102}$ such that L is a restriction of L':

$$L\left(\frac{P(z)}{Q(z)}\right) = \tilde{L}\left(\frac{P(z) \cdot H(z)}{Q(z) \cdot H(z)}\right)$$

where $(P, Q) \in \mathscr{A}_{3,2}$, $H(z) = 1 + h_1 z + h_2 z^2 + \dots + h_{100} z^{100}$

$\sigma(L) = \sigma(L') = 15$. Hence $\tilde{L} = \tilde{L}(a_1, a_2, \ldots, a_{15}; b_1, \ldots, b_{15})$.

$$\begin{cases} a_1 = c_1 + b_1 \\ a_2 = c_2 + c_1 b_1 - b_2 \\ \ldots \\ a_{15} = c_{15} + c_{14} b_1 + c_{13} b_2 + \cdots + c_1 b_{14} - b_{15} \\ \ldots \end{cases}$$

where

$$\frac{P(z)}{Q(z)} = \frac{1 + a_1 z + a_2 z^2 + \cdots + a_{103} z^{103}}{1 + b_1 z + b_2 z^2 + \cdots + b_{102} z^{102}} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots$$

$$\tilde{L} = \tilde{L}(c_1, c_2, \ldots, c_{15}, b_1, b_2, \ldots, b_{15}).$$

If $(P(z), Q(z)) \in \mathscr{A}_{3,2}$ then there exists $H_0(z) = 1 + \cdots + h_{100} z^{100}$ such that

$$Q(z) H_0(z) = 1 + o(z^{15})$$

We have

$$(P, Q) \in \mathscr{A}_{3,2}\ L(P, Q) = \tilde{L}(PH_0, QH_0) = \tilde{L}(PH_0, 1 + o(z^{15})) =$$

$$\tilde{L}(c_1, c_2, \ldots, c_{15}, b_1, b_2, \ldots, b_{15})\big|_{b_1 = b_2 = \cdots = b_{15} = 0} = L^*(c_1, c_2, \ldots, c_{15}).$$

$L = L\left(\frac{P(z)}{Q(z)}\right)$, $\frac{P(z)}{Q(z)} \in A_{2|2}$. Consider $\frac{P(z)}{Q(z)} \in A_{3|3}$.

$$\frac{P(z)}{Q(z)} = \frac{1+a_1z+a_2z^2+a_3z^3}{1+b_1z+b_2z^2+b_3z^3} = 1+\left(\frac{p_1z}{1+\mu_1z}+\frac{p_2z}{1+\mu_2z}+\frac{p_3z}{1+\mu_3z}\right)$$

$$L'\left(\frac{P(z)}{Q(z)}\right) = L'\left(1+\left(\frac{p_1z}{1+\mu_1z}+\frac{p_2z}{1+\mu_2z}+\frac{p_3z}{1+\mu_3z}\right)\right) =$$

$$L\left(1+\left(\frac{p_1z}{1+\mu_1z}+\frac{p_2z}{1+\mu_2z}\right)\right)+\{(1,2)\to(1,3)\}+\{(1,2)\to(2,3)\}$$

$$-L\left(1+\left(\frac{p_1z}{1+\mu_1z}\right)\right)-L\left(1+\left(\frac{p_2z}{1+\mu_2z}\right)\right)-L\left(1+\left(\frac{p_3z}{1+\mu_3z}\right)\right)$$

$$+L(1)$$

$$L'\left(\frac{P(z)}{Q(z)}\right) = L'(a_1, a_2, a_3; b_1, b_2, b_3) = \frac{A(a_1, a_2, a_3; b_1, b_2, b_3)}{B(a_1, a_2, a_3; b_1, b_2, b_3)}$$

$$\frac{A(a_1, a_2, a_3; b_1, b_2, b_3)}{B(a_1, a_2, a_3; b_1, b_2, b_3)} = L(a_1, a_2, a_3; b_1, b_2, b_3) \text{ if } \mathrm{Res}\,(P, Q) = 0$$

$$\frac{A}{B} \to \frac{A + G\mathrm{Res}\,(P, Q)}{B} = L$$

THANK YOU