Thick morphisms and higher Koszul brackets

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The talk is based on the work with Ted Voronov

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Papers that talk is based on are

[1] H.M.Khudaverdian, Th. Voronov *Higher Poisson brackets and differential forms*, 2008a In: Geometric Methods in Physics. AIP Conference Proceedings 1079, American Institute of Physics, Melville, New York, 2008, 203-215., arXiv: 0808.3406

[2] Th. Voronov, *Nonlinear pullback on functions and a formal category extending the category of supermanifolds]*, arXiv: 1409.6475

[3] Th. Voronov, *Microformal geometry*, arXiv: 1411.6720

Abstract...

For an arbitrary manifold M, we consider supermanifolds ΠTM and ΠT^*M, where Π is the parity reversion functor. The space ΠT^*M possesses canonical odd Schouten bracket and space ΠTM posseses canonical de Rham differential d. An arbitrary even function P on ΠT^*M such that $[P, P] = 0$ induces a homotopy Poisson bracket on M, a differential, d_P on ΠT^*M, and higher Koszul brackets on ΠTM. (If P is fiberwise quadratic, then we arrive at standard structures of Poisson geometry.) Using the language of Q-manifolds and in particular of Lie algebroids, we study the interplay between canonical structures and structures depending on P. Then using just recently invented theory of thick morphisms we construct a non-linear map between the L_∞ algebra of functions on ΠTM with higher Koszul brackets and the Lie algebra of functions on ΠT^*M with the canonical odd Schouten bracket.

Poisson manifold

Let M be Poisson manifold with Poisson tensor $P = P^{ab}\partial_b \wedge \partial_a$

$$\{f,g\} = \{f,g\}_P = \frac{\partial f}{\partial x^a} P^{ab} \frac{\partial g}{\partial x^b}\,.$$

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0\,,$$

$$\Updownarrow$$

$$P^{ar}\partial_r P^{bc} + P^{br}\partial_r P^{ca} + P^{cr}\partial_r P^{ab} = 0\,.$$

If P is non-degenerate, then $\omega = (P^{-1})_{ab} dx^a \wedge dx^b$ is closed non-degenerate form defining symplectic structure on M.

Differentials

d—de Rham differential, $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$,

$$d^2 = 0\,, df = \frac{\partial f}{\partial x^a} dx^a\,, \qquad d(\omega \wedge \rho) = d\omega \wedge \rho + (-1)^{p(\omega)} \omega \wedge d\rho\,,$$

d_P—Lichnerowicz- Poisson differential, $d_P\colon \mathfrak{A}^k(M) \to \mathfrak{A}^{k+1}(M)$,

$$d_P f = \frac{\partial f}{\partial x^b} P^{ba} \frac{\partial}{\partial x^a} \text{ (for a function } f = f(x))\,, d_P^2 = 0\,,$$

$$d_P P = 0 \leftrightarrow \text{Jacobi identity for odd Poisson bracket } [\,,\,]$$

Differential forms and multivector fields

$\mathfrak{A}^*$ space of multivector fields on M,
Ω^* space of differential forms on M,

$$\begin{array}{ccc} \mathfrak{A}^k(M) & \xrightarrow{d_P} & \mathfrak{A}^{k+1}(M) \\ \uparrow & & \uparrow \\ \Omega^k(M) & \xrightarrow{d} & \Omega^{k+1}(M) \end{array}$$

Differential forms and multivector fields

$\mathfrak{A}^*$— multivector fields on M= functions on ΠT^*M
Ω^*— differential forms on M= functions on ΠTM,

$$
\begin{array}{ccccccc}
\mathfrak{A}^k(M) & \xrightarrow{d_P} & \mathfrak{A}^{k+1}(M) & \qquad & C(\Pi T^*M) & \xrightarrow{d_P} & C(\Pi T^*M) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\Omega^k(M) & \xrightarrow{d} & \Omega^{k+1}(M) & & C(\Pi TM) & \xrightarrow{d} & C(\Pi TM)
\end{array}
$$

$$d\omega(x,\xi) = \xi^a \frac{\partial}{\partial x^a}\omega(x,\xi)\,, d_P F(x,\theta) = [P,F]_1\ ,$$

$[P,F]_1$-canonical odd Poisson bracket on ΠT^*M.

$x^a = (x^1, \ldots, x^n)$— coordinates on M
$(x^a, \xi^b) = (x^1, \ldots, x^n; \xi^1, \ldots, \xi^n)$, —coordinates on ΠTM

$$p(\xi^a) = p(x^a) + 1, x^{a'} = x^{a'}(x^a) \to \xi^{a'} = \xi^a \frac{\partial x^{a'}}{\partial x^a} . \qquad (dx^a \leftrightarrow \xi^a) .$$

‘ Respectively
$(x^a, \theta_b) = (x^1, \ldots, x^n; \theta_1, \ldots, \theta_n)$, —coordinates on $\Pi T^* M$

$$p(\theta_a) = p(x^a) + 1, x^{a'} = x^{a'}(x^a) \to \theta_{a'} = \theta_a \frac{\partial x^a}{\partial x^{a'}} . \qquad (\partial_a \leftrightarrow \theta_a) .$$

‘

Example

$$\Omega^* \ni \omega = l_a dx^a + r_{ab} dx^a \wedge dx^b \leftrightarrow \omega(x, \xi) = l_a \xi^a + r_{ab} \xi^a \xi^b \in C(\Pi TM)$$

$$\mathfrak{A}^* \ni F = X^a \partial_a + M^{ab} \partial_a \wedge \partial_b \leftrightarrow F(x, \theta) = X^a \theta_a + M^{ab} \theta_a \theta_b \in C(\Pi T^* M) .$$

Canonical odd Poisson bracket

F, G multivector fields
$[F, G]$ Schouten commutator'

F, G functions on $\Pi T^* M$
$[F, G]$ odd Poisson bracket'

$\mathbf{X} = X^a \partial_a, [\mathbf{X}, F] = \mathfrak{L}_{\mathbf{X}} F$
$P = P^{ab} \partial_a \wedge \partial_b\,, [P, F] = d_P F$'

$[\mathbf{X}, F] = [X^a \theta_a, F(x, \theta)]$
$d_P F = [P, F] = [P^{ab} \theta_a \theta_b, F(x, \theta)]$

$$[F(x,\theta), G(x,\theta)] = \frac{\partial F(x,\theta)}{\partial x^a} \frac{\partial G(x,\theta)}{\partial \theta_a} + (-1)^{p(F)} \frac{\partial F(x,\theta)}{\partial \theta_a} \frac{\partial G(x,\theta)}{\partial x^a} \,.$$

Names are

odd Poisson bracket
Schouten bracket
Buttin bracket
anti-bracket

Koszul bracket on differential forms

$$\begin{array}{c}\varphi_P\colon \Pi T^*M \to \Pi TM\\ \varphi_P^*\colon C(\Pi T^*M) \leftarrow C(\Pi TM)\end{array}, \qquad \xi^a = P^{ab}\theta_b \,\text{or}\, dx^a = P^{ab}\partial_b$$

From bracket $[\,,\,]$ on functions to Koszul bracket on diff. forms

$$[\omega,\sigma]_P = (\varphi_P^*)^{-1}\left([\varphi_P^*(\omega),\varphi_P^*(\sigma)]_P\right) .$$

$$[f,g]_P = 0\,,\, [f,dg]_P = (-1)^{p(f)}\{f,g\}_P\,,\, [df,dg]_P = (-1)^{p(f)}d\left(\{f,g\}_P\right)$$

This formula survives the limit if P is degenerate.

Question

$$\text{We have} \qquad \Pi T^* M \xrightarrow{\varphi_P} \Pi TM$$

What happens if even function $P = \frac{1}{2}P^{ab}(x,\theta)\theta_a\theta_b$ is replaced by an arbitrary even function $P = P(x,\theta)$ which obeys the master-equation

$$[P,P] = 2\frac{\partial P(x,\theta)}{\partial x^a}\frac{\partial P(x,\theta)}{\partial \theta^a} = 0\,.$$

(In the case $P = \frac{1}{2}P^{ab}(x,\theta)\theta_a\theta_b$ master-equation is just Jacobi identity for Poisson bracket $\{\,,\}_P$ on M.)

Master-Hamiltonian $\to$ brackets (I-st case)

M—(super)manifold. (coordinates '$x = x^a$)

Odd Hamiltonian $Q(x,p)$ on T^*M, ($p = p_b$ fibre coordinates) defines homotopy odd Poisson (Schouten) brackets on M— collection $\{\{\}_Q, \{\,,\,\}_Q, \{\,,\,,\,\}_Q, \dots\}$ of brackets on M:

$$\{f\}_Q = (Q,f)\Big|_{p=0}, \quad \{f,g\}_Q = ((Q,f),g)\Big|_{p=0},$$

$$\{f_1,f_2,\dots,f_n\}_Q = ((\dots(Q,f_1),f_2),\dots,f_n)\Big|_{p=0}$$

$(\,,\,)$—canonical even Poisson bracket on T^*M.
$(Q,Q) = 0$—Jacobi identity,

We come to usual odd Poisson bracket if Hamiltonian is quadratic in momenta, $Q = Q^{ab}p_ap_b$.

Master-Hamiltonian $\to$ brackets (II-nd case)

Even Hamiltonian $H(x,\theta)$ on ΠT^*M, ($\theta = \theta_b$ fibre coordinates)
defines homotopy Poisson brackets on M—
collection $\{\{\}_H, \{\,,\,\}_H, \{\,,\,,\,\}_H, \dots\}$ of brackets on M:

$$\{f\}_H = [H,f]\big|_{\theta=0}, \quad \{f,g\}_H = [[H,f]\,,g]\,\big|_{\theta=0},$$

$$\{f_1,f_2,\dots,f_n\}_H = [[\dots[H,f_1]\,,f_2]\,,\dots,f_n]\,\big|_{\theta=0}$$

$[\,,\,]$—canonical odd Poisson bracket on ΠT^*M
$[H,H]=0$— Jacobi identity
We come to usual even Poisson bracket if Hamiltonian is quadratic in momenta, $H = H^{ab}\theta_a\theta_b$.

Mackenzie-Xu symplectomorphism

$E \to B$—vector bundle. Canonical symplectomorphism (MX-symplectomorphism)

$$T^*E \leftrightarrow T^*E^*$$

Local coordinates

$$\underbrace{\overbrace{x^\mu, u^i}^{\text{coord. on } E} \; ; p_\mu, p_j}_{\text{coord. on } T^*E}, \qquad \underbrace{\overbrace{y^\nu, u_i}^{\text{coord. on } E^*} \; ; q_\mu, p^k}_{\text{coord. on } T^*E^*}.$$

Then $\kappa\colon\ T^*E \to T^*E^*$ is such that

$$\kappa^*(y^\mu) = x^\mu\,,\ \kappa^*(u_i) = p_i\,,\ \kappa^*(q_\mu) = -p_\mu\,,\ \kappa(p^i) = u^i\,.$$

Canonical odd Poisson bracket on $\Pi T^* M$

Consider an odd Hamiltonian $Q = p_a \eta^a$ on tangent bundle $T^*(\Pi T^* M)$ to $\Pi T^* M$.

$$\text{coordinates} \qquad \underbrace{\overbrace{x^a, \theta_b}^{\Pi T^* M}; p_a, \eta^b}_{T^*(\Pi T^* M)}$$

Odd Hamiltonian $Q = p_a \eta^a$ is quadratic in momenta.
It generates an odd canonical Poisson bracket $[\,,\,]$ on $\Pi T^* M$:
$[f, g] = [f, g]_P = ((Q, f), h) =$

$$= \left(\eta^a \frac{\partial f}{\partial x^a} + p_a \frac{\partial g}{\partial \theta_a}, g \right) = \frac{\partial f}{\partial x^a} \frac{\partial g}{\partial \theta_a} + \frac{\partial g}{\partial \theta_a} \frac{\partial f}{\partial x^a}$$

$(\,,\,)$ is canonical Poisson bracket on $T^*(\Pi T^* M)$.

Consider MX symplectomorphism $T^*(\Pi T^*M) \leftrightarrow T^*(\Pi TM)$:

$$\underbrace{\overbrace{x^a, \theta_b}^{\Pi T^* M}; p_a, \eta^b}_{T^*(\Pi T^* M)} \leftrightarrow \underbrace{\overbrace{x^a, \xi^b}^{\Pi TM}; q_a, \pi_b}_{T^*(\Pi TM)}$$

$p_a \leftrightarrow -q_a$, $\theta_b \leftrightarrow \pi_{\mathbf{b}}$, $\eta^a \leftrightarrow \xi^a$,
Odd Hamiltonian $Q = p_a\eta^a \leftrightarrow$ odd Hamiltonian $K = q_a\xi^a$.

$$[\omega] = (K, \omega) = \xi^a \frac{\partial\omega}{\partial x^a} = d\omega, (\omega(x,\xi) \to \omega(x, dx)).$$

(all higher brackets vanish)
Odd homtopy bracket is nothing but de Rham differential.

Lichnerowicz differential d_P

For even function $P = P(x, \theta)$ ($[P, P] = 0$)

$$d_P F = [P, F] . (d_P^2 = 0)$$

Consider

$$Q_P = (P, Q) = (P, p_a \eta^a) = p_a \frac{\partial P(x, \theta)}{\partial \theta_a} + \eta^a \frac{\partial P(x, \theta)}{\partial x^a}$$

This is Hamiltonian linear in momenta. It produces degenerate homotopy bracket—Lichnerowicz differential:.

$$[F] = (Q_P, F) = ((P, Q), F) = [P, F] = d_P F.$$

(all higher brackets vanish)

Lichnerowic differential → Higher Koszul brackets

Under MX symplectomorphism, Hamiltonian

$$Q_P(x,\theta,p,\eta) = p_a \frac{\partial P(x,\theta)}{\partial \theta_a a} + \eta^a \frac{\partial P(x,\theta)}{\partial x^a} \text{ on } T^*(\Pi T^* M)$$

transforms to Hamiltonian

$$K_P(x,\xi,q,\pi) = q_a \frac{\partial P(x,\pi)}{\partial \pi_a a} + \eta^a \frac{\partial P(x,\pi)}{\partial x^a} \text{ on } T^*(\Pi TM)\,.$$

This Hamiltonian defines homotopy Schouten bracket on ΠTM (Higher Koszul bracket on differential forms)

Higher Koszul brackets on M

Odd Hamiltonian K_P on $T^*(\Pi TM)$ defines homotopy odd Poisson bracket (higher Koszul bracket) on ΠTM,

$$[F_1, F_2, \dots, F_n]_P = [\dots[K_P, F_1], \dots, F_p]\big|_{\Pi TM}, \qquad \big|_{\Pi TM} = \big|_{p=\pi=0}.$$

$$F = F(x,\xi) = f(x) + \xi^a f_a(x) + \dots, (df = \xi^a \partial_a f),$$

$$[f]_P = 0, [f_1, f_2, \dots, f_k]_P = 0$$

$$[f_1, df_2, \dots, df_n] = \{f_1, f_2, \dots, f_n\},$$

$$[df_1, df_2, \dots, df_n] = d\{f_1, f_2, \dots, f_n\},$$

In the same way as for classical case ($P = P^{ab}\theta_b\theta_a$)

Recall the classical case $P = P^{ab}\theta_b\theta_a$

$\mathfrak{A}^*$— multivector fields on M= functions on ΠT^*M
Ω^*— differential forms on M= functions on ΠTM,

$$
\begin{array}{ccccccc}
\mathfrak{A}^k(M) & \xrightarrow{d_P} & \mathfrak{A}^{k+1}(M) & \qquad & C(\Pi T^*M) & \xrightarrow{d_P} & C(\Pi T^*M) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\Omega^k(M) & \xrightarrow{d} & \Omega^{k+1}(M) & & C(\Pi TM) & \xrightarrow{d} & C(\Pi TM)
\end{array}
$$

$$\varphi_P\colon\, \xi^a = P^{ab}\theta_b\,, \quad \varphi_P^*\colon\, C(\Pi TM) \to C(\Pi T^*M)$$

Then

$$\varphi_P^*(d\omega) = d_P(\varphi_P^*\omega)\,.$$

This relation survives for an arbitrary $P = P(x,\theta)$ ($[P,P] = 0$.)

Two Hamiltonians

$$\varphi_P\,\Pi TM \to \Pi T^*M\colon\ \xi^a = \frac{1}{2}\frac{\partial P(x,\theta)}{\partial \theta_a}\,.$$

$$\varphi_P^*(d\omega) = d_P(\varphi_P^*\omega).,\qquad (Q_P, \varphi_P^*\omega) = \varphi_P^*((K,\varphi))\,.$$

$$\begin{array}{ccc}
T^*(\Pi T^*M) & \longrightarrow & T^*(\Pi TM) \\
Q = p_a\eta^a & \longrightarrow & K = \eta^a q_a \\
\text{canonical odd bracket} & \longrightarrow & \text{de Rham differential on } \Pi TM \\
Q_P = (P, Q) & \longrightarrow & K_P \\
\text{Lichnerowicz diff. on } \Pi T^*M & \longrightarrow & \text{Higher Koszul bracket on } \Pi TM
\end{array}$$

Map φ_P intertwines Lichnerowicz and de Rham differentials, i.e. Hamiltonians Q_P and K.

Question

Map φ_P intertwines Lichnerowicz and de Rham differentials, i.e. Hamiltonians Q_P and K.

How look a map which intertwines Hamiltonians, Q and K_P, i.e. a map which intewins canonical Schouten bracket and higher Koszul brackets???

Usual Poisson bracket

$P(x,\theta)=P^{ab}(x)\theta_a\theta_b$ even function on ΠT^*M quadratic on θ defines usual Poisson bracket on M: for $f,g\in C(M)$

$$\{f,g\}=\{f(x),g(x)\}_P=[[P,f],g]=\frac{\partial}{\partial\theta^b}\left(\frac{\partial P(x,\theta)}{\partial\theta^a}\frac{\partial f}{\partial x^a}\right)\frac{\partial g}{\partial x^b}=$$

$$\frac{\partial f}{\partial x^a}P^{ab}\frac{\partial g}{\partial x^b}\,.$$

$$\text{Jacobi identity}: 0=[P,P]=2\frac{\partial P}{\partial x^a}\frac{\partial P}{\partial\theta_a}=4\partial_a P^{bc}P^{ad}\theta_b\theta_c\theta_d$$

i.e.

$$P^{da}\partial_a P^{bc}+P^{ba}\partial_a P^{cd}+P^{ca}\partial_a P^{db}=0\,.$$

Higher Poisson brackets on M

Even (non-quadratic in momenta) Hamiltonian in ΠT^*M, $H = P(x, \theta)$, ($[P, P] = 0$ Jacobi identity) defines homotopy Poisson brackets, higher even brackets:

$$\{f_1, f_2, \ldots, f_n\}_P = [\ldots[P, f_1], \ldots, f_p]\big|_M, \qquad \big|_M = \big|_{\theta=0}.$$

If

$$P = P^a\theta_a + \frac{1}{2}P^{ab}\theta_b\theta_a + \frac{1}{6}P^{abc}\theta_c\theta_b\theta_a + \ldots$$

then

$$\{x^a\}_P = P^a, \{x^a, x^b\} = P^{ab}, \{x^a, x^b, x^c\} = P^{abc} \ldots,$$

From ΠT^*M to ΠTM.

Theorem

There is a natural odd linear map $C(\Pi T^*M) \to C(T^*(\Pi TM))$ that takes canonical odd Poisson bracket on ΠT^*M to canonical even Poisson bracket on $T^*(\Pi TM)$

Corollary

$$\begin{array}{ccc} \textit{even Hamiltonian} & & \textit{odd Hamiltonian} \\ P = P(x,\theta) \textit{ on } \Pi T^*M & \longrightarrow & K = K_P(x,\xi;p,\pi) \textit{ on } T^*(\Pi TM) \\ \textit{defining higher even} & & \textit{defining higher Koszul} \\ \textit{Poisson bracket on } M & & \textit{bracket on } \Pi T^*M \end{array}$$

Recall: Master-Hamiltonian $\to$ homotopy brackets

M—(super)manifold. (coordinates $x = x^a$)

Odd Hamiltonian $Q(x,p)$ on T^*M, ($p = p_b$ fibre coordinates) defines homotopy odd Poisson (Schouten) brackets on M—collection $\{\{\}_Q, \{\,,\,\}_Q, \{\,,\,,\,\}_Q, \dots\}$ of brackets on M:

$$\{f\}_Q = (H,f)\big|_{p=0}\,, \quad \{f,g\}_H = ((H,f)\,,g)\big|_{p=0}\,,$$

$$\{f_1,f_2,\dots,f_n\}_Q = (\dots(Q,f_1),f_2),\dots,)\big|_{p_b=0}$$

$(\,,\,)$—canonical even Poisson bracket on T^*M.
$(Q,Q) = 0$—Jacobi identity,

We come to usual odd Poisson bracket if Hamiltonian is quadratic in momenta, $Q = Q^{ab}p_a p_b$.

Homotopy bracket on $M \to L_\infty$-algebra of functions on $M \to Q$-manifold

M an arbitrary (super)manifold

Let $Q = Q(x, p)$ be an odd Hamiltonian in T^*M, and Jacobi identity $(Q, Q) = 0$ is obeyed.
The odd Hamiltonian Q defines homotopy odd Poisson (homotopy Schouten) bracket on M.
iConisder the following Hamilton-Jacobi vector field

$$\mathbf{X}_Q\colon\ C(T^*M) \ni f \to f + \varepsilon Q\left(x^a, p_b = \frac{\partial f(x)}{\partial x^b}\right),$$

$$\mathbf{X}_Q = \int_M dxQ\left(x^a, \frac{\partial f(x)}{\partial x^b}\right)\frac{\delta}{\delta f(x)}, \mathbf{X}_Q^2 = \frac{1}{2}[\mathbf{X}_Q, \mathbf{X}_Q] = 0.$$

$\mathbf{X}_Q$ is homological vector field on infinite-dimensional space $\mathfrak{M} = C(M)$ of functions on manifold M.

Homotopy Schouten structure on functions on M
defined by odd Hamiltonian Q

$$\downarrow$$

Q-manifold $(\mathfrak{M}, \mathbf{X}_Q)$, L_∞ algebra
$\mathfrak{M} = C(M)$ and $\mathbf{X}_Q$ is Hamilton Jacobi field of Q

$$P = P(x,\theta),\ [P,P] = 0.$$

$$\begin{array}{ccc} \Pi T^*M - (x^a, \theta_b) & & \Pi TM - (x^a, \xi^b) \\ \text{Odd Poisson canonical bracket} & & \text{Odd homotopy Koszul bracket} \\ \text{Hamiltonian } Q = p_a \xi^a & \rightarrow & \text{Hamiltonian } K_P = \xi^a \frac{\partial P}{\partial x^a} + p_a \frac{\partial P}{\partial \theta_a} \\ \text{on } T(\Pi T^*M) - (x^a, \theta_b; p_a, \xi^a) & & \text{on } T^*(\Pi TM) - (y^a, \xi^b; p_a, \theta_b) \end{array}$$

$$\begin{array}{cc} Q\text{-manifold, } L_\infty \text{ algebra} & Q\text{-manifold, } L_\infty \text{ algebra} \\ \mathfrak{M}_1 = C(\Pi T^*M)\,, \mathbf{X}_1 = \mathbf{X}_Q & \mathfrak{M}_2 = C(\Pi TM)\,, \mathbf{X}_2 = \mathbf{X}_{K_p} \end{array}$$

Does there exist L_∞-morphism $(\mathfrak{M}_2, \mathbf{X}_2) \to (\mathfrak{M}_1, \mathbf{X}_1)$, i.e. map $\mathfrak{M}_2 \to \mathfrak{M}_1$ (may be non-linear) which intertwines homological vector fields $\mathbf{X}_1, \mathbf{X}_2$?

Theorem

Yes, it does.

Special case, $P = \frac{1}{2}P^{ab}\theta_b\theta_a$

In this case the map

$$\Pi T^*M \to \Pi TM\colon \quad \xi^a = \frac{\partial P}{\partial \theta^a} = P^{ab}(x)\theta_b\,,$$

is linear in fibres. Morphism of Q-manifolds

$$C(\Pi T^*M) \leftarrow C(\Pi TM)$$

is its pull-back.
These linear maps intertwine differentials d and d_P, Hamiltonians Q and K_P and their homological vector fields $\mathbf{X}_Q$ and $\mathbf{X}_{K_p}$ on infinite-dimensional spaces of functions.

It is more tricky if $P(x,\theta)$ is an arbitrary function (solution of master-equation $[P,P]=0$. The map

$$\Pi T^*M \to \Pi TM\colon \quad \xi^a = \frac{\partial P(x,\theta)}{\partial \theta^a}$$

is in general non-linear map.
Does there exist morphism of Q-manifolds $(\mathfrak{M}_2,\mathbf{X}_2) = (C(\Pi TM),\mathbf{X}_{K_P}) \to (\mathfrak{M}_1,\mathbf{X}_1) = (C(\Pi T^*M),\mathbf{X}_Q)$?
In other words does there exist a (non-linear) map $C(\Pi TM) \to C(\Pi T^*M)$? which intertwines canonical odd Poisson bracket $[\,,\,]$ on ΠT^*M and homotopy Koszul brackets $[\,]_P, [\,,\,]_P, [\,,\,,\,]_P \ldots$ on ΠTM?

Recall: Two Hamiltonians

$$\varphi_P \,\Pi TM \to \Pi T^* M\colon\ \xi^a = \frac{1}{2}\frac{\partial P(x,\theta)}{\partial \theta_a}\,.$$

$$\varphi_P^*(d\omega) = d_P(\varphi_P^*\omega).,\qquad (Q_P, \varphi_P^*\omega) = \varphi_P^*((K,\varphi))\,.$$

$T^*(\Pi T^* M)$	$\longrightarrow$	$T^*(\Pi TM)$
$Q = p_a\eta^a$	$\longrightarrow$	$K = \eta^a q_a$
canonical odd bracket	$\longrightarrow$	de Rham differential on ΠTM
$Q_P = (P, Q)$	$\longrightarrow$	K_P
Lichnerowicz diff. on $\Pi T^* M$	$\longrightarrow$	Higher Koszul bracket on ΠTM

Map φ_P intertwines Lichnerowicz and de Rham differentials, i.e. Hamiltonians Q_P and K.

Recall:Question

Map φ_P intertwines Lichnerowicz and de Rham differentials, i.e. Hamiltonians Q_P and K.

How look a map which intertwines Hamiltonians Q and K_P? a map which intertwines canonical Schouten bracket and higher Koszul brackets???

Answer

Morphism $\varphi_P \colon \Pi T^*M \to \Pi TM$ intertwines Hamiltonians Q_p and K
We try to construct a 'morphism', (sort of morphism)

$$\Phi \colon \Pi T^*M \to \Pi TM\,,$$

which intertwines Hamiltonians Q and K_P

$\Phi = \phi_P^*$ is *thick morphism* which is adjoint to morphism φ_P.

Definition of thick morphism. (T.Voronov)

M_1, M_2–two (super)manifolds
x^i–coordinates on M_1, y^a–coordinates on M_2

Consider symplectic manifold $T^*M_1 \times (-T^*M_2)$
equipped with canonical symplectic structure

$$\omega = \omega_1 - \omega_2 = \underbrace{dp_i \wedge dx^i}_{\text{coord. on } T^*M_1} - \underbrace{dq_a \wedge dy^a}_{\text{coord. on } T^*M_2}$$

function $S = S(x, q)$
defines Lagrangian surface $\Lambda_S \subset T^*M_1 \times (-T^*M_2)$:

$$\Lambda_S = \left\{ (x, p, y, q) : \quad p_i = \frac{\partial S(x,q)}{\partial x^i}, y^b = \frac{\partial S(x,q)}{\partial q_b} \right\}$$

Lagrangian surface—canonical relation—thick morphism

Lagr. surf. Λ_S is canon. relation Φ_s in $T^*M_1 \times (-T^*M_2)$

$$(x^i, p_j) \sim_S (y^a, q_b) \leftrightarrow (x^i, p_j, y^a, q_b) \in \Lambda_S\,, (\Phi_S = \sim_S)\,.$$

$$\Phi = \Phi_s \text{is a thick morphism } M_1 \Rrightarrow M_2$$

It defines pull-back Φ_S^* of functions

$$\Phi_S^*\colon\ \mathfrak{M}_2 = C(M_2) \to \mathfrak{M}_1 = C(M_1)\,,$$

such that for every function $g = g(y) \in \mathfrak{M}_2$,

$$f = f(x) = (\Phi_S^* g)(x)\colon\ \Lambda_f = \Phi_S \circ \Lambda_g\,,$$

where Λ_f, Λ_g are Lagrangian surfaces, graphs of df, dg in T^*M_1, T^*M_2.

Explicit expression

$$f(x) = (\Phi_S^* g)(x) = g(y) + S(x,q) - y^a q_a$$

where y^a and q_a are defined from the equations

$$y^a = \frac{\partial S(x,q)}{\partial q_a}\,, \quad q_a = \frac{\partial g(y)}{\partial y^a}$$

We see that $\Lambda_f = \Phi_S \circ \Lambda_g$ since

$$p_i = \frac{\partial f}{\partial x^i} = \frac{\partial}{\partial x^i}\left(g(y) + S(x,q) - y^a q_a\right) = \frac{\partial S(x,q)}{\partial x^i}\,.$$

Properties of thick morphism

Example

Generating function $S = S^a(x) q_a$

$$(\Phi_S^* g)(x) = g(y) + S(x,q) - y^a q_a = g(y) + \underbrace{(S^a(x) - y^a)}_{\text{vanishes}} q_a = g(S^a(x))$$

Thick morphism $M_1 \overset{\Phi_s}{\Rrightarrow} M_2$ is usual morphism $M_1 \overset{y^a = S^a(x)}{\rightarrow} M_2$.
In general case if $S(x,q) = S(x) + S^a(x) q_a q_b + S^{ab}(x) q_a q_b + \dots$

$$(\Phi_S^* g)(x) = S(x) + \left(g(y) + S^{ab}(x) \frac{\partial g(y)}{\partial y^a} \frac{\partial g(y)}{\partial y^b} + \dots \right)_{y^a = S^a(x)}$$

is non-linear pull-back.

Why it is important. Voronov's Theorem and Corollary

Theorem

Let $\Phi_S \colon M_1 \Rrightarrow M_2$ be a thick morphism.
Let Q_1, Q_2 be Φ_S related Hamilt. on T^*M_1, T^*M_2: :

$$Q_1\left(x^i, p_j = \frac{\partial S(x,q)}{\partial x^j}\right) \equiv Q_2\left(y^a = \frac{\partial S(x,q)}{\partial q_a}, q_b\right).$$

Then Hamilton-Jacobi vector fields $\mathbf{X}_{Q_1}, \mathbf{X}_{Q_2}$ on spaces $\mathfrak{M}_1, \mathfrak{M}_2$ of functions are related by non-linear pull back Φ_S^*

Corollary

Let $\Phi_S\colon M_1 \Rrightarrow M_2$ be a thick morphism.
If odd Hamiltonians $Q_1.Q_2$ are Φ_S related and

$$(Q_1, Q_1) = (Q_2) = Q_2) = 0\,,$$

then non-linear pull-back

$$\Phi_S^*\colon \mathfrak{M}_2 \to \mathfrak{M}_1$$

defines L_∞-morphism of L_∞ algebras $(\mathfrak{M}_1, \mathbf{X}_{Q_1})$ and $(\mathfrak{M}_2, \mathbf{X}_{Q_2})$.

Revenons à nos moutons

$$\varphi_P \,\Pi TM \to \Pi T^* M\colon\ \xi^a = \frac{1}{2}\frac{\partial P(x,\theta)}{\partial \theta_a}\,.$$

$$\varphi_P^*(d\omega) = d_P(\varphi_P^*\omega)\,.,\qquad (Q_P, \varphi_P^*\omega) = \varphi_P^*((K,\varphi))\,.$$

$T^*(\Pi T^* M)$	$\longrightarrow$	$T^*(\Pi TM)$
$Q = p_a \eta^a$	$\longrightarrow$	$K = \eta^a q_a$
canonical odd bracket	$\longrightarrow$	de Rham differential on ΠTM
$Q_P = (P, Q)$	$\longrightarrow$	K_P
Lichnerowicz diff. on $\Pi T^* M$	$\longrightarrow$	Higher Koszul bracket on ΠTM

Morphism (usual) φ_P intertwines Hamilt. Q_P and K
??? thick morphism ϕ_S which intertwines Hamilt. Q and K_P.

Q_P and K are φ_P related, Q and K_P will be ϕ_P related

Thick morphism—generalisation of adjoint

$$
\begin{array}{ccc}
E \to M \text{ vector bundle} & & E^* \to M \text{ its dual} \\
\Phi\colon E \Rrightarrow E^* & \leftrightarrow & \Phi^+\colon E \Rrightarrow E^* \\
L = L_S \text{ Lagr. surf.defining } \Phi & \leftrightarrow & L^* = L_{S^*} \text{ Lagr. surf.defining } \Phi`
\end{array}
$$

These Lagrangian surfaces belong to $T^*E \times (-T^*E^*$.

$$
\begin{array}{ccc}
T^*E \times (-T^*E^*) & \overset{\text{MX symplectom.}}{\leftrightarrow} & T^*E \times (-T^*E^*) \\
S & \leftrightarrow & S^* \\
\Lambda = \Lambda_S & \leftrightarrow & \Lambda^* = \Lambda_{S^*}
\end{array}
$$

If Φ is linear map in fibres, then Φ^* is just its adjoint.

Return again to our case

$$\varphi_P\, \Pi TM \to \Pi T^*M\colon\ \xi^a = \frac{1}{2}\frac{\partial P(x,\theta)}{\partial \theta_a}\,.$$

$$\varphi_P^*(d\omega) = d_P(\varphi_P^*\omega)\,., \qquad (Q_P, \varphi_P^*\omega) = \varphi_P^*((K,\varphi))\,.$$

$$\begin{array}{ccc}
T^*(\Pi T^*M) & \longrightarrow & T^*(\Pi TM) \\
Q = p_a\eta^a & \longrightarrow & K = \eta^a q_a \\
\text{canonical odd bracket} & \longrightarrow & \text{de Rham differential on } \Pi TM \\
Q_P = (P,Q) & \longrightarrow & K_P \\
\text{Lichnerowicz diff. on } \Pi T^*M & \longrightarrow & \text{Higher Koszul bracket on } \Pi TM
\end{array}$$

Morphism (usual) φ_P intertwines Hamilt. Q_P and K
??? thick morphism ϕ_S ? which intertwines Hamilt. Q and K_P.

Q_P and K are φ_P related, Q and K_P will be ϕ_P related

Solution

$$\varphi_P\colon\qquad \Pi TM \to \Pi T^*M\colon\ \xi^a = \frac{1}{2}\frac{\partial P(x,\theta)}{\partial \theta_a}\,.$$

Let ϕ be a thick morphism adjoint to morphism φ_P.
We know that φ_P intertwines Q_P and K
We have that Mackenzie-Xu symplectomorphism transforms:

$$\varphi_P \leftrightarrow \phi$$
$$K \leftrightarrow Q$$
$$Q_P \leftrightarrow K_P$$

Hence adjoint thick morphism ϕ intertwines Q and K_P.
The pull-back $\phi^*\colon\ C(\Pi T^*M) \leftarrow C(\Pi TM)$ is non-linear map on space of functions which transfors homotopy Koszul brcket to canonical Schouten bracket.

Thank you

Papers that talk is based on

[1] H.M.Khudaverdian, Th. Voronov *Higher Poisson brackets and differential forms*, 2008a In: Geometric Methods in Physics. AIP Conference Proceedings 1079, American Institute of Physics, Melville, New York, 2008, 203-215., arXiv: 0808.3406

[2] Th. Voronov, *Nonlinear pullback on functions and a formal category extending the category of supermanifolds]*, arXiv: 1409.6475

[3] Th. Voronov, *Microformal geometry*, arXiv: 1411.6720