Compensating field in odd Laplacian and modular class

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Contents

Abstract...

A second order operator Δ can be uniquely defined by its principal symbol S and potential U, if it acts on half-densities. The potential U is a second order compensating field, (second order connection). It compensates (gauges) the action of diffeomorphisms on the second derivatives in an operator Δ in the same way as an affine connection compensates the action of diffeomorphisms on first derivatives in the first order operator, a covariant derivative.

...Abstract

We consider cases of Riemannian and odd Poisson supermanifolds. If an even principal symbol S defines Riemannian structure, then one can uniquely define compensating field U via Levi-Civita connection of Riemannian metric. There is no Levi-Civita connection in a case if an odd principal symbol S defines an odd Poisson structure on supermanifold. On the other hand one can naturally consider in this case modular class taking values in first Poisson cohomology. In the case if this class vanishes then one can define the compensating field by the condition $\Delta^2 = 0$. We consider examples including symplectic case. At the end we disucss results of Klaus Bering and Igor Batalin.

Let $\mathbf{S} = S^{ab}\partial_b \otimes \partial_a$ be a rank two contravariant symmetric tensor field on manifold M.
Assign to this field a second order operator

$$\mathbf{S} = S^{ab}\partial_b \otimes \partial_a \mapsto \quad \Delta = S^{ab}(x)\partial_b\partial_a + \ldots$$

which contains not much additional data...

Computational experiment

Consider operator acting on densities of weight λ:

$$\Delta\colon \Psi(x)|Dx|^{\lambda} \mapsto$$

$$\left(S^{ik}(x)\partial_k\partial_i\Psi(x) + p\partial_k S^{ki}(x)\partial_i\Psi(x) + U(x)\Psi(x)\right)|Dx|^{\lambda}$$

where p, λ are parameters
Try to fix parameters p, λ such that under changing of coordinates operator preserves its form.

Computational experiment...

Under changing of coordinates $x^i = x^i(x^{i'})$

$$\Psi(x')|Dx'|^\lambda = \Psi(x'(x))J^\lambda_{(x',x)}|Dx|^\lambda\,, \left(J_{(x',x)} = \det\left(\frac{\partial x'}{\partial x}\right)\right)\,.$$

Computational experiment...

Under changing of coordinates $x^i = x^i(x^{i'})$

$$\Psi(x')|Dx'|^\lambda = \Psi(x'(x))J^\lambda_{(x',x)}|Dx|^\lambda\,, \left(J_{(x',x)} = \det\left(\frac{\partial x'}{\partial x}\right)\right)\,.$$

$$\Delta\Psi = \Delta\left(\Psi(x'(x))J^\lambda_{(x',x)}|Dx|^\lambda\right) =$$

$$\left(S^{ik}(x)\partial_k\partial_i\left(\Psi(x')J^{\lambda}_{(x',x)}\right)+p\partial_k S^{ki}(x)\partial_i\left(\Psi(x')J^{\lambda}_{(x',x)}\right)+\dots\right)\frac{|Dx'|^{\lambda}}{J^{\lambda}_{(x',x)}}=$$

$$\left(S^{i'k'}(x')\partial_{k'}\partial_{i'}\Psi(x')+p\partial_{k'}S^{k'i'}(x)\partial_{i'}\Psi(x')+\dots\right)|Dx'|^{\lambda}+$$

$$+\left(\underbrace{(1-p)S^{i'k'}\partial_{k'}\partial_{i'}\Psi(x')+(2\lambda-p)\partial_i\left(\log\det J_{(x',x)}\right)S^{ik}\partial_k\Psi}_{\text{undesirable terms}}\right)|Dx'|^{\lambda}$$

Put

$$p=1\,,\qquad \lambda=\frac{p}{2}=\frac{1}{2}\,.$$

We come to operator $\Delta=\Delta_S$ on half-densities:

Δ-Operator on half-densities

$$S^{ik}(x) \mapsto \Delta\colon\ \Delta\left(\Psi(x)\sqrt{Dx}\right) =$$

$$\left(S^{ik}(x)\partial_k\partial_i\Psi(x) + \partial_k S^{ki}(x)\partial_i\Psi(x) + U(x)\Psi(x)\right)\sqrt{Dx} =$$

$$\left(\partial_k\left(S^{ik}(x)\partial_i\Psi(x)\right) + U(x)\Psi(x)\right)\sqrt{Dx} =$$

We assign to tensor field S an operator which is defined up to a function $U(x)$.

Global definition of operator Δ corresponding to S

$$S^{ik}(x) \mapsto \Delta:$$

- $$\Delta = S^{ik}\partial_k\partial_i + \ldots,,$$
  i.e. principal symbol of the operator is tensor field $S^{ik}(x)$,
- operator Δ is self-adjoint:

$$\Delta^* = \Delta, \qquad \langle \Delta\Psi_1, \Psi_2\rangle = \langle \Psi_1, \Delta^*\Psi_2\rangle,$$

$$\langle \Psi_1\sqrt{Dx}, \Psi_2\sqrt{Dx}\rangle = \int_M \Psi_1(x)\Psi_2(x)|Dx|.$$

Two self-adjoint operators Δ_1, Δ_2 with the same principal symbol differ on a scalar function, $\Delta_1 - \Delta_2 =$ scalar function

Does there exist at least one operator obeying the conditions above? Yes, it does

Let $\rho(x)|Dx|$ be an arbitrary volume form on M. An arbitrary half-density $\Psi(x)\sqrt{Dx}$ defines vector field

$$\mathbf{D}_\Psi = D^i_\Psi(x)\frac{\partial}{\partial x^i} = S^{ik}(x)\partial_k\left(\frac{\Psi(x)}{\sqrt{\rho(x)}}\right)\partial_i\,,$$

Consider an operator $\Delta_{S,\rho}$ such that

$$\Delta_{S,\rho}(\Psi\sqrt{Dx}) = \sqrt{\rho}\left(\mathrm{div}_\rho\mathbf{D}_\Psi\right)\sqrt{Dx} = \sqrt{\rho}\frac{1}{\rho}\frac{\partial}{\partial x^i}\left(\rho D^i_\Psi\right)\sqrt{Dx} =$$

$$\frac{1}{\sqrt{\rho}}\frac{\partial}{\partial x^i}\left(\rho(x)S^{ik}(x)\frac{\partial}{\partial x^k}\left(\frac{\Psi(x)}{\sqrt{\rho(x)}}\right)\right)\sqrt{Dx} =$$

$$\left(\partial_k\left(S^{ki}(x)\partial_i\Psi(x)\right)+U_\rho(x)\Psi(x)\right)\sqrt{Dx}\,,$$

where

$$U_\rho(x)=-\frac{1}{4}\partial_i\log\rho\, S^{ik}\partial_k\log\rho-\frac{1}{2}\partial_i\left(S^{ik}\partial_k\log\rho\right)\,.$$

Any self-adjoint operator Δ on half-densities with principal symbol S^{ik} differs from the operator $\Delta_{S,\rho}$ on a scalar function:

$$\Delta=\Delta_{S,\rho}+F\,.$$

Construction for global operator without volume form

One may construct operator without using volume form:

$$\mathbf{S}(x) = \mathbf{X}_\lambda(x) \otimes_{\mathrm{sym}} \mathbf{Y}_\lambda(x), \quad S^{ik}(x) = \frac{1}{2}\sum_\lambda \left(X^i(x)_\lambda Y^k_\lambda(x) + Y^i_\lambda(x) X^k_\lambda(x) \right)$$

$$S^{ik}(x) \mapsto \Delta = \frac{1}{2}\sum_\lambda \left(\mathscr{L}_{\mathbf{X}_\lambda} \mathscr{L}_{\mathbf{Y}_\lambda} + \mathscr{L}_{\mathbf{Y}_\lambda} \mathscr{L}_{\mathbf{X}_\lambda} \right),$$

where $\mathscr{L}_\mathbf{X}$ Lie derivative of half-density along vector field $\mathbf{X}$:

$$\mathscr{L}_\mathbf{X} : \quad \mathscr{L}_\mathbf{X}(\Psi\sqrt{Dx}) = \left(X^i(x)\partial_i \Psi(x) + \frac{1}{2}\partial_i X^i(x)\Psi(x) \right) \sqrt{Dx}.$$

Δ is self-adjoint operator with principal symbol **S**.

Changing of coordinates

$$\mathbf{s} = \Psi(x)\sqrt{|Dx|} \mapsto \Delta\mathbf{s} = \left(\partial_a\left(S^{ab}\partial_b\Psi(x)\right) + U(x)\right)\sqrt{|Dx|}\,.$$

In new coordinates

$$\mathbf{s} = \Psi'(x')\sqrt{|Dx'|}\,,\ \Psi'(x') = \Psi(x(x'))\sqrt{\left|\det\left(\frac{\partial x}{\partial x'}\right)\right|}\,.$$

$$\mathbf{s} = \Psi'(x')\sqrt{|Dx'|} \mapsto \Delta\mathbf{s} = \left(\partial_{a'}\left(S^{a'b'}\partial_{b'}\Psi'(x')\right) + U'(x')\right)\sqrt{|Dx'|}\,,$$

Principal symbol $\mathbf{S} = S^{ab}(x)\partial_b\partial_a$ is a tensor,

$$S^{a'b'} = \frac{\partial x^{a'}}{\partial x^a}\frac{\partial x^{b'}}{\partial x^b}S^{ab}$$

How U transforms?

Transformation of potential

Potential U transforms in the following way

$$U'(x') = U(x) + \frac{1}{2}\partial_a\left(S^{ab}\partial_b \log J\right) + \frac{1}{4}\partial_a \log J S^{ab}\partial_b \log J\,,$$

where

$$J = \det\left(\frac{\partial x'}{\partial x}\right)\,,$$

$$\partial_a \log J = \frac{\partial^2 x^{a'}}{\partial x^a \partial x^b}\frac{\partial x^b}{\partial x^{a'}}\,.$$

Potential U— second order compensation field

Usual connection is ‘first order’ compensating field:

$$\partial_{\mathbf{Y}} = Y^r(x)\frac{\partial}{\partial x^r} \longrightarrow \nabla_{\mathbf{Y}} = Y^r(x)\left(\frac{\partial}{\partial x^r} + \Gamma^i_{rm}(x)\right),$$

Γ^i_{rm} are Christoffel symbols of affine connection. Christoffel symbols are first order compensating fields,

$$\Gamma'^{i'}_{k'm'} = \frac{\partial x^{i'}}{\partial x^i}\Gamma^i_{km}\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}} + \frac{\partial x^{i'}}{\partial x^r}\frac{\partial^2 x^r}{\partial x^{k'}\partial x^{m'}}.$$

Potential U on manifold equipped with volume form

Consider the following simple but important example.

Let manifold M be equipped with a volume form $\rho = \rho(x)|Dx|$.

$$
\begin{array}{ccc}
\text{volume form } \rho = \rho(x)|Dx| & & \text{local coordinates } x^a\colon\ \rho = |Dx| \\
\Downarrow & & \Downarrow \\
U = U_\rho(x) & & U \equiv 0
\end{array}
$$

$$
\left(U_\rho(x) = -\tfrac{1}{4}\partial_i \log\rho\, S^{ik}\partial_k \log\rho - \tfrac{1}{2}\partial_i\left(S^{ik}\partial_k \log\rho\right)\right)
$$

In ‘unimodular’ local coordinates x^a, $\rho = |Dx|$ and

$$
\mathbf{s} = \Psi\sqrt{Dx}\,, \quad \Delta\mathbf{s} = \partial_a\left(S^{ab}\partial_b\Psi(x)\right)\sqrt{Dx}\,.
$$

First order connection on densities

Let $\mathbf{s} = s(x)|Dx|^\lambda$, $\mathbf{X} = X^a\partial_a$.

Consider

$$\nabla_{\mathbf{X}}\mathbf{s} = \nabla_{\mathbf{X}}\left(\Psi(x)|Dx|^\lambda\right) = X^a\left(\partial_a\Psi(x) + \lambda\gamma_a(x)\Psi(x)\right)|Dx|^\lambda\,,$$

$$\gamma_a\colon\ \gamma_a Dx = \nabla_a Dx\,,$$

γ_a—-first order connection on densities,

$$\gamma_{a'} = \frac{\partial x^a}{\partial x^{a'}}\left(\partial_a \log J + \gamma_a\right)\,, \quad \left(\log J = \det\left(\partial x'\partial x\right)\right)\,.$$

From first order connection γ_a to second order compensating field U

Principal symbol $\mathbf{S}$ and the connection γ_a define a pencil of second order operators on densities of arbitrary weight λ:

$$\Delta_\lambda : \mathbf{s} = \Psi(x)|Dx|^\lambda \mapsto \Delta_\lambda \mathbf{s} = \mathrm{div}_\gamma(\mathbf{S}\nabla\mathbf{s}) =$$

$$\left(\partial_b\left(S^{ba}\partial_a\Psi\right) + (2\lambda-1)\gamma^a\partial_a\Psi + \lambda\partial_a\gamma^a + \lambda(\lambda-1)\gamma^a\gamma_a\Psi\right)|Dx|^\lambda\,,$$

where $\gamma^a = S^{ab}\gamma_b$. (H.K., T.T. Voronov (2003), [11]))

$$\text{for half densities, } \lambda = \tfrac{1}{2}\,, \Delta = \partial_a\left(S^{ba}\partial_a\ldots\right) + \frac{1}{2}\partial_a\gamma^a - \frac{1}{4}\gamma^a\gamma_a\,.$$

$$\text{We see that } (\mathbf{S},\gamma_a(x)) \mapsto U = \frac{1}{2}\partial_a\gamma^a - \frac{1}{4}\gamma^a\gamma_a$$

Connection on densities induced by volume form

A volume form $\rho = \rho(x)|Dx|$ induces $\nabla_{\mathbf{X}}$:

$$\mathbf{s} = \Psi|Dx|^{\lambda} \mapsto \nabla_{\mathbf{X}}(\mathbf{s}) = X^a \partial_a \left(\frac{\mathbf{s}}{\rho^{\lambda}}\right) \rho^{\lambda} = X^a (\partial_a \Psi - \lambda \partial_a \log \rho \Psi) |Dx|^{\lambda},$$

$$\gamma_a = -\partial_a \log \rho .$$

Connection on densities induced by affine connection

Affine connection on vector fields
$(\nabla^{\mathrm{affine}}_{\mathbf{X}}\mathbf{Y} = X^c\left(\partial_c Y^a + \Gamma^a_{cb} Y^b\right)\partial_a)$ induces

$$\nabla_{\mathbf{X}}(\mathbf{s}) = X^a\left(\partial_a s(x) - \Gamma^c_{ac} s(x)\right)|Dx|^{\lambda},$$

$$\gamma_a = -\Gamma^c_{ac}.$$

Levi-Civita connection on vector fields and connection on densites

Let M be Riemannian manifold $G = g_{ab}dx^a dx^b$,

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} \left(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc} \right)$$

Levi-Civita connection (unique symmetric connection preserving metric)
Then

$$\gamma_a = -\Gamma^c_{ac} = -\partial_a \log \rho^{(G)}$$

where $\rho^{(G)} = \sqrt{\det g_{ab}}|Dx|$ invariant volume form.

$$\text{Canonical operator } \Delta \mathbf{s} = \Big(\partial_a (g^{ab} \partial_b \Psi(x)) + U(x)\Psi(x) \Big) \sqrt{Dx} ,$$

where $U = \frac{1}{2}\partial_a \gamma^a - \frac{1}{4}\gamma^a \gamma_a$.

Levi-Civita-like statement for second order compensation field U

Manifolds with distingushed volume form possess canonical second order connection, potential U

$$U \equiv 0 \text{ in coordinates } x \text{ such that } \rho = |Dx|$$

In arbitrary coordinates

$$U = \frac{1}{2}\partial_a \gamma^a - \frac{1}{4}\gamma_a \gamma^a, \quad (\gamma_a = -\partial_a \log \rho).$$

We see that in particular Riemannian manifold possesses Levi-Civita-like second order compensating field U.

Canonical operator on ΠTN

Consider supermanifold $M = \Pi TN$, tangent bundle over usual manifold N with reversed parity of fibres
$\{q^i\}$-local ooordinates on $N \Rightarrow \{q^i, \xi^j\}$-local coordinates on ΠTN

$$p(\xi^j) = 1 \quad \{q^i\} \to \{q^{i'}\}, \xi^{j'} = \xi^i \frac{\partial q^{j'}(x)}{\partial q^i} .$$

Manifold ΠTN possesses canoncail volume form
$Dq = D(x,\xi) = dx^1 \dots dx^n d\xi^1 \dots d\xi^n$, $D(x,\xi) = D(x',\xi')$.
Canonical second order compensating field $U(x,\xi) = 0$.

Fact Every second order rank 2 tensor field $\mathbf{S}$ on ΠTN defines canonical second order operator on half-densities

$$\Delta_{\mathbf{S}} = \partial_a \left(S^{ab}(q,\xi) \partial_b \Psi(q,\xi) \right) \sqrt{D(x,\xi)}, \quad (x^a = (q^i, \xi^j))$$

Example of projective connection

Consider on **R** operator: $\Delta = (\partial_x^2 + U(x))|Dx|^2$,

$$\Psi(x)|Dx|^{-1/2} \to \Delta\Psi = (\Psi_{xx} + U(x)\Psi(x))\,|Dx|^{3/2}$$

Under changing of coordinates $y = y(x)$,

$$\Delta = (\partial_x^2 + U(x))|Dx|^2 \mapsto (\partial_y^2 + \tilde{U}(y))|Dy|^2\,,$$

where

$$\tilde{U}(y) = U(y(x)) + \underbrace{\frac{1}{2}\left(\frac{x_{yyy}}{x_y} - \frac{3}{2}\left(\frac{x_{yy}}{x_y}\right)^2\right)}_{\text{Schwarzian of transformation } x = x(y)}$$

In the second part of the talk we consder the case when there is no distinguished volume form, and there is no distinguished first order connection

In the first part of the talk we ignored the difference between manifolds and supermanifolds, in particular we omitted sign factors $(-1)^{\cdots}$. Now we will be much more carefull.

Odd operator

Let $\mathbf{E} = E^{ab}\partial_b\partial_a$ be an odd symmetric rank 2 tensor field:

$$E^{ab} = (-1)^{p(b)p(a)}E^{ba}, \quad p(E^{ab}) = 1 + p(a) + p(b)$$

One can consider odd Laplace operator

$$\Delta = E^{ab}\partial_b\partial_a + \dots$$

$z^a = (\underbrace{x^i}_{\text{even}}, \underbrace{\theta^\alpha}_{\text{odd}})$ even and odd coordinates on supermanifold

$$\partial_a\partial_b = \frac{\partial}{\partial z^a}\frac{\partial}{\partial z^b} = (-1)^{p(a)p(b)}\frac{\partial}{\partial z^b}\frac{\partial}{\partial z^a} = (-1)^{p(a)p(b)}\partial_b\partial_a$$

Proposition

Let $\mathbf{E} = E^{ab}\partial_b \otimes \partial_a$ be symmetric rank 2 odd tensor field, let $\Delta \in \mathscr{F}_{\mathbf{S}}$ be an arbitrary self-adjoint operator on half-densities with principal symbol $\mathbf{E}$, $\Delta = E^{ab}\partial_b\partial_a + \dots$. Then

- Δ^2 is anti-self-adjoint operato: $(\Delta^2)^* = -\Delta^2$.
- order of the operator Δ^2 is equal to 3 or 1 or $\Delta^2 = 0$.

Proof.

$$\Delta^* = \Delta\,, \quad p(\Delta) = 1\,;$$

$$\left(\Delta^2\right)^* = \left(\frac{1}{2}[\Delta, \Delta]\right)^* = \frac{1}{2}(\Delta\Delta + \Delta\Delta)^* = (\Delta\Delta)^* = (-\Delta^*\Delta^*) = -\Delta^2\,.$$

Operator Δ^2 is a commutor of second order odd operator with itself, hence its order is less or equal to 3.

□

...Proof. Order of operator Δ^2

Δ^2 is anti-self-adjoint operator, of the order ≤ 3.

Show that order of Δ^2 cannot be equal neither to 2 nor to 0.

if $\Delta^2 = L^{ab}\partial_b\partial_a + \ldots$ then $(\Delta^*)^2 = -\Delta = L^{ba}\partial_b\partial_a$ hence $L^{ab} \equiv 0$.
if $\Delta^2 = F(x)$ then $(\Delta^*)^2 = -\Delta = -F(x)$, hence $F(x) \equiv 0$.

Hence order of operator Δ^2 can be equal to 3, to 1 or $\Delta^2 = 0$.

Bracket on supermanifold—- Odd Poisson manifold

Consider supermanifold M provided with an odd symmetric tensor field E^{ab}

$$E^{ab} = (-1)^{p(a)p(b)} E^{ba} \qquad p(E^{ab}) = 1 + p(a) + p(b)$$

It defines an odd bracket:

$$\{f, g\} = (-1)^{p(f)p(a)} \partial_a f E^{ab} \partial_b g \,,$$

$$\{f, g\} = -(-1)^{(p(f)+1)(p(g)+1)} \{g, f\} \,,$$

What about Jacobi identity?

$$\{f, \{g, h\}\} = \{\{f, g\}, h\} + (-1)^{p(f)p(g)} \{g, \{f, h\}\} \quad \text{Jacobi identity}$$

Odd Δ-operator—-odd Poisson manifold

$$\Delta = E^{ab}\partial_b\partial_a + \dots, \quad p(\Delta) = 1$$

$$\Delta^2 = \frac{1}{2}[\Delta, \Delta] = K^{abc}\partial_c\partial_b\partial_a + \dots$$

$$\mathbf{K} = [\mathbf{E}, \mathbf{E}], \quad \text{Schoutten commutator}$$

$$K^{abc} = \left((-1)^{a+ac}E^{ad}\partial_d E^{bc} + \text{cyclic permutations}\right)$$

$\mathbf{K} = [\mathbf{E}, \mathbf{E}] \equiv 0 \Leftrightarrow$ Jacoby identities for $\{f, g\} = (-1)^{p(f)p(a)}\partial_a f E^{ab}\partial_b g$,

order of Δ^2 less than $3 \Leftrightarrow (M, \mathbf{E})$ is an odd Poisson supermanifold

On the other hand we showed that if order of Δ^2 is less than 3 then order of Δ^2 is ≤ 1.

Odd Poisson manifold in terms of Δ-operator

Let **E** be rank 2 symmetric odd tensor field on manifold M, and let $\Delta = E^{ab}\partial_b\partial_a + \ldots$ be an arbitrary self-adjoint odd operator on half-density with principla symbol **E**, ($\Delta \in \mathscr{F}_{\mathbf{E}}$).

$(M, \mathbf{E})$ is an odd Poisson manifold $\Leftrightarrow$ $\Delta^2 = \mathscr{L}_{\mathbf{X}}$ is an operator of order ≤ 1

Modular class of an odd Poisson manifold

Theorem

Let $(M, \mathbf{E})$ be an odd Poisson manifold, $\{f, g\} = (-1)^{fa}\partial_a f E^{ab}\partial_b g$. Let $\mathscr{F}_{\mathbf{E}}$ be a class of self-adjoint second order odd operators on half-densities with principal symbol $\mathbf{E}$. Then

$$\forall \Delta \in \mathscr{F}_{\mathbf{E}}, \Delta \mapsto \mathbf{X} = \mathbf{X}_\Delta : \ \Delta^2 = \mathscr{L}_{\mathbf{X}_\Delta},$$

where $\mathbf{X}$ is Poisson vector field on M.

$$\forall \Delta_1, \Delta_2 \in \mathscr{F}_{\mathbf{E}}, \Delta_2 = \Delta_1 + F, \mathbf{X}_2 = \mathbf{X}_1 + D_F,$$

where D_F is Hamiltonian vector field: $D_F G = \{F, G\}$.
Odd Poisson manifold $\rightarrow$ modular class of vector fields.

This class was introduced by H.M.Kh and Voronov already in [9], but in terms of operator on functions.

Modular class of (usual) Poisson manifold(recalling)

Let $(M, \mathbf{E})$ be an (usual) Poisson manifold ($p(\mathbf{E}) = 0$).
Choose an arbitrary volume form ρ and consider vector field

$$\mathbf{X}_\rho : \hat{\mathbf{X}}_\rho f = \mathrm{div}_{\,\rho} D_f = \frac{1}{\rho}\partial_a \left(\rho E^{ab}\partial_b f\right) .$$

If $\rho' = e^H \rho$ is another volume form, then

$$\mathbf{X}_{\rho'} = \mathrm{div}_{\,\rho'} D_f = \frac{1}{e^H \rho}\partial_a \left(e^H \rho E^{ab}\partial_b F\right) = \mathbf{X}_\rho + D_H$$

Definition(Weinstein 1994) Modular class of Poisson manifold is an equivalence class of vector field $[X_\mu]$ modulo Hamiltonian vector fields.
(Modular class is an element in the first Poisson cohomology group.)

Example

Let Poisson manifold be symplectic, i.e. tensor **E** is non-degenerate. Choose Lioville volume form,

$$\rho = \sqrt{\det(E_{ab})} Dx \,, (\rho = D(q,p) \text{ in Darboux coordinates } (q^i, p_j)$$

$\mathbf{X}_\rho = 0$, modular class $[\mathbf{X}]$ vanishes $\Leftrightarrow$ Lioville Theorem

Example

Let $\mathscr{G}$ be Lie algebra. $\{u_i, u_k\} = c_{ik}^m u_m$.
Choose volume form $\rho = Du = du_1 \dots du_n$

$$\operatorname{div}_\rho D_f = \frac{\partial}{\partial u_i}\left(u_m c_{ik}^m \frac{\partial f(u)}{\partial u_k}\right) = c_{mk}^m \frac{\partial}{\partial u_k} f$$

Modular class is just modular vector field $\mathbf{X} = c_{mk}^m \frac{\partial}{\partial u_k}$

Modular class for odd Poisson bracket—-Batalin-Vilkovisky operator

Even Poisson structure	—	Odd Poisson structure
	$f \mapsto \mathrm{div}_\rho D_f$	
first order operator		second order operator

E.g. for odd symplectic manifold in Darboux coordinates (x^a, θ_b), $(\{x^a, \theta_b\} = \delta^a_b$, $\{x^a, x^b\} = \{\theta_a, \theta_b\} = 0)$ we come to the famous Batalin-Vilkovisky operator (1981)

$$\Delta_\rho f = \frac{1}{2}\mathrm{div}_\rho D_f = \frac{\partial^2 f}{\partial x^a \partial \theta_a}$$

if we choose $\rho = |D(x, \theta|$ (H.Kh, 1989 [7]).

Return to an odd operator on half-densities.

$$\mathscr{F}_{\mathbf{E}} \ni \Delta,\ \Delta^2 = \mathscr{L}_{\mathbf{X}}, \quad \Delta \to \Delta + F, \mathbf{X} \to \mathbf{X} + D_F$$

,

Odd Poisson supermanifold $(M, \mathbf{E}) \mapsto$ modular class $[\mathbf{X}]\colon\ [\Delta^2] = [\mathbf{X}]$

$$[\mathbf{X}] = 0 \Leftrightarrow \exists \Delta \in \mathscr{F}_{\mathbf{E}}\colon \quad \Delta^2 = 0\,.$$

Indeed let $\Delta^2 = \mathbf{X}$. Since $[\mathbf{X}] = 0$, hence $\mathbf{X} = -D_F$. Hence

$$(\Delta + F)^2 = 0\,.$$

Example of non-vanishing modular class

On $(m|1)$-dimensional supermanifold consider two vector fields $\underbrace{\mathbf{A}=K^i\partial_i+Y\partial_\theta}_{\text{even}}$ and $\underbrace{\mathbf{B}=\partial_\theta}_{\text{odd}}$.

Consider second order self-adjoint operator on half-densities

$$\Delta=\frac{1}{2}\left(\mathscr{L}_{\mathbf{A}}\circ\mathscr{L}_{\mathbf{B}}+\mathscr{L}_{\mathbf{B}}\circ\mathscr{L}_{\mathbf{A}}\right).$$

$$\begin{matrix}\text{Jacobi identity} & & \text{Order of operator } \Delta^2\\ \text{for odd Poisson bracket} & \Leftrightarrow & \text{is less than 3}\\ \{x^i,\theta\}\}=K^i(x,\theta) & & \Delta^2=\mathscr{L}_{\mathbf{X}}\end{matrix}$$

$$\Updownarrow$$

$$K^i\partial_\theta K^j+K^j\partial_\theta K^i\equiv 0\Leftrightarrow\begin{cases}K^i=F^i(x),\Delta^2=0,\\ K^i=\theta\eta^i(x),\Delta^2=\eta^2=\eta^m\partial_m\eta^i=\frac{1}{2}[\eta,\eta]\end{cases}$$

Example of non-vanishing modular class

Coordinates

$$(\underbrace{x^m}_{even}, \underbrace{\eta^i_m, \theta}_{odd})$$

$$\{x^i, \theta\} = -\{\theta, x^i\} = x^m \theta \eta^i_m$$

(all other brackets vanish)
Then

$$\Delta^2 = \mathscr{L}_{\mathbf{X}}$$

where

$$[\mathbf{X}] = [x^n \eta^m_n \eta^i_m \partial_i] .$$

Symplectic case

Suppose Poisson tensor E^{ab} is non-degenerate. (This implies that manifold M is $(n|n)$-dimensional.) There is no distinguished volume form for an odd symplectic structure, but

Theorem

On odd sympelctic supermanifold there exists canonical operator on half-densities $\Delta^{\#}$:

$$\Delta^{\#}\left(s(x,\theta)\sqrt{D(x,\theta)}\right)=\frac{\partial^2 s(x,\theta)}{\partial x^i\partial\theta_i}\sqrt{D(x,\theta)}$$

in Darboux coordinates (x^i,θ_j) ($\{x^i,\theta_j\}=\delta^i_j$, $\{x^i,x^j\}=0$, $\{\theta_i,\theta_j\}=0$), (H.Kh. 1999,[8])

Potential $U=0$ in Darboux coordinates, and

$$\left(\Delta^{\#}\right)^2=0,\quad i.e.\mathbf{X}_{\Delta^{\#}}=0\,.$$

Potential *U* as a unique solution of differential equation

In arbitrary coordinates canonical operator

$$\Delta^{\#} = E^{ik}\partial_k\partial_i + \partial_k E^{ki}\partial_i + U_{can}\,,,$$

$$U_{\rm can}: \qquad \left(\Delta^{\#}\right)^2 = \mathscr{L}_{\mathbf{X}} = 0$$

This is firsts order differential equation:

$$\mathbf{X} = X^q\partial_q: \;\; X^q = \partial_b(E^{ba}\partial_a\partial_p E^{pq}) + 2(-1)^{p(q)}E^{qb}\partial_b U = 0\,.$$

which has unique solution (if we put odd constant to 0) since tensor E^{ab} is non-degenerate (symplectic case).

General case of an odd Poisson manifold

In the special case if Poisson manifold contains symplectic leaves: Coordinates $(x^i, \theta_i, z^\alpha)$ such that

$$\{x^i, \theta_j\} = \delta^i_j \text{ all other brackets for coordinates vanish}$$

Then $\Delta = \frac{\partial^2}{\partial x^i \partial \theta_i}$ is invariant operator on half-densities.

???

Bering's formulae

K. Bering in 2006 wrote the formula for operator $\Delta^{\#}$ in arbitrary coordinates. He comes to the answer

$$U = \frac{1}{4}\partial_m\partial_n E^{nm} - \frac{1}{24}\partial_i E^{mn} e_{np} E^{pi} \qquad (*)$$

He already wrote in 2007 the answer for Poisson case in terms of tensor E and e, where $\mathbf{EeE} = \mathbf{E}$ [5].
For both answers it was checked that the expression is invariant with respect to infinitesimal changing of coordinates.

Batalin-Bering formulae

2) Let $\rho|Dx|$ be an arbitrary volume form on odd symplectic manifold, and G be an arbitrary odd Riemannian structure compatible with the volume form, then the scalar curvature of the Riemannian structure is proportional to

$$\frac{\Delta^{\#}\left(\sqrt{\rho|Dx|}\right)}{\sqrt{\rho|Dx|}}\,.$$

This formula was obtained by I. Batalin and K. Bering in [1]

'Existence' of answer in general coordinates and uniqueness of connection

Existence of Levi-Civita connection
i.e. the unique symm. afine connection $\Rightarrow$
compatible with Riemannian structure

Existence of a formu
expressing this coonec
in terms of metric and its de

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} \left(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc} \right)$$

There are MANY[1]
affine connections
compatible with
symplectic structure

$\Rightarrow$

there is no a formula
expressing at least one
of these coonections
in terms of metric and its derivatives.

---

[1]one may take a connections such that its Chrsitoffels vanish in given Darboux coordinates

'Non-existence' of formula for sympelctic connection

Proof.

Suppose

$$\Gamma^a_{bc} = \Gamma^a_c \left(E^{pq}, \partial_c E^{pq}, \dots\right)$$

Hence in Darboux coordinates Γ^a_{bc} are constants.
Contradiction. □

Uniqueness of potential $U \Rightarrow$ Existence of formula $U = U(E, \partial E, \partial^2 E)$

Calculating U in arbitrary coordinates we come to Bering's formula.

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