Potential in canonical odd Laplacian

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Contents

Abstract...

Second order operator can be uniquely defined by its principal symbol S, $\Delta = S\partial^2 + U$, if it acts on half-densities. A geometrical object, the potential U, can be considered as a second order connection,in a sense that it is a compensating (gauge) field, which compensates the action of diffeomorphisms on the operator Δ, in the same way as an affine connection compensates the action of diffeomorphisms on vector fields in the first order operator, covariant derivative, $\nabla = \partial + \Gamma$. The geometrical object, potential U, can be canonically assigned to affine connection. Thus one comes to a canonical Laplacian on half-densities in Riemannian manifold, the potential corresponding to the Levi-Civita connection of a Riemannian metric.

...Abstract

In odd symplectic manifold, one can consider second order operators with principal symbol defined by odd symplectic structure, but there is no unique affine connection compatible with symplectic structure. On the other hand one can define a unique potential U in this case. We deliver explicit constructions for the potential U in an odd symplectic supermanifold coming to an expression for the canonical odd second order operator in arbitrary coordinates.

Let

$$S^{ik}(x) = S^{ki}(x)$$

be a rank two contravariant symmetric tensor field on manifold M.
Assign to this field a second order operator

$$\Delta = S^{ik}(x)\partial_k\partial_i + \ldots$$

which contains not much additional data...

Computational experiment

Consider operator acting on densities of weight λ:

$$\Delta\colon \Psi(x)|Dx|^{\lambda} \mapsto$$

$$\left(S^{ik}(x)\partial_k\partial_i\Psi(x) + p\partial_k S^{ki}(x)\partial_i\Psi(x) + U(x)\Psi(x)\right)|Dx|^{\lambda}$$

where p, λ are parameters
Try to fix parameters p, λ such that under changing of coordinates operator preserves its form.

Computational experiment...

Under changing of coordinates $x^i = x^i(x^{i'})$

$$\Psi(x')|Dx'|^{\lambda} = \Psi(x'(x))J^{\lambda}_{(x',x)}|Dx|^{\lambda}\,, \left(J_{(x',x)} = \det\left(\frac{\partial x'}{\partial x}\right)\right)\,.$$

Computational experiment...

Under changing of coordinates $x^i = x^i(x^{i'})$

$$\Psi(x')|Dx'|^\lambda = \Psi(x'(x))J^\lambda_{(x',x)}|Dx|^\lambda\,, \left(J_{(x',x)} = \det\left(\frac{\partial x'}{\partial x}\right)\right)\,.$$

$$\Delta\Psi = \Delta\left(\Psi(x'(x))J^\lambda_{(x',x)}|Dx|^\lambda\right) =$$

$$\left(S^{ik}(x)\partial_k\partial_i\left(\Psi(x')J^{\lambda}_{(x',x)}\right)+p\partial_k S^{ki}(x)\partial_i\left(\Psi(x')J^{\lambda}_{(x',x)}\right)+\dots\right)\frac{|Dx'|^{\lambda}}{J^{\lambda}_{(x',x)}}=$$

$$\left(S^{i'k'}(x')\partial_{k'}\partial_{i'}\Psi(x')+p\partial_{k'}S^{k'i'}(x)\partial_{i'}\Psi(x')+\dots\right)|Dx'|^{\lambda}+$$

$$+\left(\underbrace{(1-p)S^{i'k'}\partial_{k'}\partial_{i'}\Psi(x')+(2\lambda-p)\partial_i\left(\log\det J_{(x',x)}\right)S^{ik}\partial_k\Psi}_{\text{undesirable terms}}\right)|Dx'|^{\lambda}$$

Put

$$p=1\,,\qquad \lambda=\frac{p}{2}=\frac{1}{2}\,.$$

We come to operator $\Delta=\Delta_S$ on half-densities:

Δ-Operator on half-densities

$$S^{ik}(x) \mapsto \Delta\colon\ \Delta\left(\Psi(x)\sqrt{Dx}\right) =$$

$$\left(S^{ik}(x)\partial_k\partial_i\Psi(x) + \partial_k S^{ki}(x)\partial_i\Psi(x) + U(x)\Psi(x)\right)\sqrt{Dx} =$$

$$\left(\partial_k\left(S^{ik}(x)\partial_i\Psi(x)\right) + U(x)\Psi(x)\right)\sqrt{Dx} =$$

We assign to tensor field S an operator which is defined up to a function $U(x)$.

Global definition of operator Δ corresponding to S

$$S^{ik}(x) \mapsto \Delta:$$

- $$\Delta = S^{ik}\partial_k\partial_i + \ldots,,$$
  i.e. principal symbol of the operator is tensor field $S^{ik}(x)$,
- operator Δ is self-adjoint:
  $$\Delta^* = \Delta\,, \qquad \langle \Delta\Psi_1, \Psi_2\rangle = \langle \Psi_1, \Delta^*\Psi_2\rangle\,,$$
  $$\langle \Psi_1\sqrt{Dx}, \Psi_2\sqrt{Dx}\rangle = \int_M \Psi_1(x)\Psi_2(x)|Dx|\,.$$

Two self-adjoint operators Δ_1, Δ_2 with the same principal symbol differ on a scalar function, $\Delta_1 - \Delta_2 =$ scalar function

Does there exist at least one operator obeying the conditions above? Yes, there does

Let $\rho(x)|Dx|$ be an arbitrary volume form on M. An arbitrary half-density $\Psi(x)\sqrt{Dx}$ defines vector field

$$\mathbf{D}_\Psi = D^i_\Psi(x)\frac{\partial}{\partial x^i} = S^{ik}(x)\partial_k\left(\frac{\Psi(x)}{\sqrt{\rho(x)}}\right)\partial_i\,,$$

Consider an operator $\Delta_{S,\rho}$ such that

$$\Delta_{S,\rho}(\Psi\sqrt{Dx}) = \sqrt{\rho}\left(\mathrm{div}_\rho\mathbf{D}_\Psi\right)\sqrt{Dx} = \sqrt{\rho}\frac{1}{\rho}\frac{\partial}{\partial x^i}\left(\rho D^i_\Psi\right)\sqrt{Dx} =$$

$$\frac{1}{\sqrt{\rho}}\frac{\partial}{\partial x^i}\left(\rho(x)S^{ik}(x)\frac{\partial}{\partial x^k}\left(\frac{\Psi(x)}{\sqrt{\rho(x)}}\right)\right)\sqrt{Dx} =$$

$$\left(\partial_k\left(S^{ki}(x)\partial_i\Psi(x)\right)+U_\rho(x)\Psi(x)\right)\sqrt{Dx}\,,$$

where

$$U_\rho(x)=-\frac{1}{4}\partial_i\log\rho\, S^{ik}\partial_k\log\rho-\frac{1}{2}\partial_i\left(S^{ik}\partial_k\log\rho\right).$$

Any self-adjoint operator Δ on half-densities with principal symbol S^{ik} differs from the operator $\Delta_{S,\rho}$ on a scalar function.

Geometrical meaning of U

$U(x)$ is compensating field.
Under change of coordinates it changes as “second order connection”
(Usual connection is ‘first order’ compensating field.)

Potential $U = U_\rho$ and connection on volume forms

$$U_\rho(x) = -\frac{1}{4}\partial_i \log\rho\, S^{ik}\partial_k \log\rho - \frac{1}{2}\partial_i\left(S^{ik}\partial_k \log\rho\right).$$

Consider $\gamma_i = -\partial_i \log\rho$, γ_i defines connection on densities:

$$\nabla_i(\Psi(x)|Dx|^\lambda) = \partial_i\left(\frac{\Psi(x)}{\rho^\lambda}\right)\rho^\lambda|Dx|^\lambda = (\partial_i\Psi(x) + \lambda\gamma_i\Psi)|Dx|^\lambda.$$

$$U = \frac{1}{2}\partial_i\left(S^{ik}\gamma_k\right) - \frac{1}{4}\gamma_i S^{ik}\gamma_k.$$

One can consider an arbitrary connection on densities ($\gamma_i \to \gamma_i + t_i$ (H.Kh., Voronov (2003) [3]).

Odd Poisson manifold

Consider supermanifold M provided with symmetric tensor field S^{ik}

$$E^{ik} = (-1)^{p(i)p(k)} E^{ki} \qquad p(S^{ik}) = 1 + p(i) + p(k)$$

It defines odd Poisson bracket:

$$\{f, g\} = (-1)^{fi} \partial_i f E^{ik} \partial_k f \,,$$

$$\{f, g\} = -(-1)^{(f+1)(g+1)} \{g, f\} \,,$$

$$\{f, \{g, h\}\} = \{\{f, g\}, h\} + (-1)^{fg} \{g, \{f, h\}\} \quad \text{Jacobi identity}$$

Jacobi identities–properties of Δ^2

$$\Delta(\Psi(x)\sqrt{Dx}) = \left(\partial_i\left(E^{ik}\partial_k\Psi(x)\right) + U(x)\Psi(x)\right)\sqrt{Dx}$$

$U(x)$ is odd potential.

$$\Delta^2 = \underbrace{\left((-1)^{i+in}E^{ir}\partial_r E^{mn} + \text{cyclic permutations}\right)}_{\text{l.h.s. of Jacobi identities}}\partial_n\partial_m\partial_i + \ldots$$

where dots denote terms of order ≤ 2,

$$\Delta^2 = \frac{1}{2}[\Delta,\Delta] = \text{operator of order} \leq 2 \qquad \Leftrightarrow (\text{Jacobi identities})\,.$$

Δ^2 is a vector field

The self-adjointness of operator Δ implies anti-self-adjointness of operator Δ^2:

$$\Delta^* = \Delta, \quad i.e. \quad \langle \Delta\Psi, \Phi\rangle = \langle \Psi, \Delta\rangle (-1)^{p(\Psi)}\,.$$

Hence

$$\langle \Delta^2\Psi, \Phi\rangle = \langle \Delta\Psi, \Delta\rangle (-1)^{p(\Psi)+1} = \langle \Psi, \Delta^2\rangle (-1)^{p(\Psi)+1+p(\Psi)} = -\langle \Psi, \Delta^2\rangle\,.$$

Hence it follows from Jacobi identities that Δ^2 is first order anti-self adjoint operator, i.e. Lie derivative (of half-densities)

$$\Delta^* = \Delta \Rightarrow (\Delta^2)^* = -(\Delta^2) \Rightarrow \Delta^2 = \mathscr{L}_{\mathbf{X}} = X^i \partial_i + \frac{1}{2}\partial_i X^i$$

If $\Delta \mapsto \Delta + F$, then

$$\Delta^2 \mapsto (\Delta + F)^2 = \Delta^2 + D_F\,,$$

where $D_F = E^{ik}\partial_k F \partial_i$ is Hamiltonian vector field. We have

$$E^{ik} \longrightarrow \{\Delta\} \longrightarrow [\mathbf{X}]_E\,,$$

where $\{\Delta\}$ is family of Δ-operators corresponding to Poisson structure E (operators differ on scalar function)
Cohomological class $[\mathbf{X}]_E$ is modular class of E.
If modular class vanishes then one can choose Δ such that $\Delta^2 = 0$.

In the work [2] this modular class was introduced in terms of operators acting on functions.

Symplectic case

Suppose E^{ik} is non-degenerate. (This implies that manifold M is $(n|n)$-dimensional.)
Then there exist canonical operator on half-densities $\Delta^{\#}$

$$\Delta^{\#}\left(\Psi(x,\theta)\sqrt{D(x,\theta)}\right) = \frac{\partial^2\Psi(x,\theta)}{\partial x^i \partial\theta_i}\sqrt{D(x,\theta)}$$

in Darboux coordinates (x^i,θ_j) ($\{x^i,\theta_j\} = \delta^i_j$, $\{x^i,x^j\} = 0$, $\{\theta_i,\theta_j\} = 0$), (H.Kh. 1999,[1])
In arbitrary coordinates canonical operator

$$\Delta^{\#} = E^{ik}\partial_k\partial_i + \partial_k E^{ki}\partial_i + U_{can}\,,,$$

where U is "second order connection" which vanishes in Darboux coordinates.

$$\left(\Delta^{\#}\right)^2 = 0, \mathbf{X}_{\Delta^{\#}} = 0\,.$$

For arbitrary operator

$$\Delta = E^{ik}\partial_k\partial_i + \partial_k E^{ki}\partial_i + U = \Delta^{\#} + F(x)\,,,$$

F is scalar function,

$$\Delta^2 = \mathscr{L}_{\mathbf{X}},$$

where $\mathbf{X} = D_F$ is Hamiltonian vector field of Hamiltonian F.

potential U—“Levi-Civita connection”

The condition $\Delta^2 = 0$ uniquely defines canonical operator $\Delta^\#$ and potential U, which is equal to zero in an (arbitrary) Darboux coordinates.

$$U = \frac{I^{mn}}{2} \times$$

$$\left(\frac{\partial^3 z^i}{\partial w^i \partial w^n \partial w^m} - \frac{\partial^2 z^i}{\partial w^j \partial w^m} \frac{\partial^2 z^j}{\partial w^i \partial w^n} + \frac{\partial^2 z^i}{\partial w^i \partial w^m} \frac{\partial^2 z^j}{\partial w^j \partial w^n} (-1)^{im+jn} \right),$$

where z^i are given coordinates and w^j are arbitrary Darboux coordinates. U does not depend on a choice of Darboux coordiantes.

Two very improtant answers

K. Bering (2006) [5], and K. Bering with I.Batalin (2008) [4], formulated the following two answers which in our terms look as following:

1) Second order connection for canonical operator is equal to

$$U = \frac{1}{4}\partial_m\partial_n E^{nm} - \frac{1}{24}\partial_i E^{mn} e_{np} E^{pi}$$

2) If $\rho|Dx|$ is an arbiotrary volume form on odd symplectic manifold, and G be an arbitrary odd Riemannian metric compatible with the volume form, then the scalar curvature is proportional to

$$\frac{\Delta^{\#}\left(\sqrt{\rho|Dx|}\right)}{\sqrt{\rho|Dx|}}.$$

These results were obtained by straightforward calculations.

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