Geometry of second order operators and odd symplectic structures

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INTEGRABLE SYSTEMS AND QUANTUM SYMMETRIES
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Contents

References

O.M. Khudaverdian, A.S. Schwarz, Yu.S. Tyupkin. *Integral invariants for Supercanonical Transformations.* Lett. Math. Phys., v. 5 (1981), p. 517-522.

References

O.M. Khudaverdian, A.S. Schwarz, Yu.S. Tyupkin. *Integral invariants for Supercanonical Transformations.* Lett. Math. Phys., v. 5 (1981), p. 517-522.

O.M. Khudaverdian, R.L. Mkrtchian. *Integral Invariants of Buttin Bracket.* Lett. Math. Phys. v. 18 (1989), p. 229-231 (Preprint EFI–918–69–86- Yerevan (1986)).

I.A.Batalin, G.A Vilkovisky. *Gauge algebra and quantization.* Phys.Lett., **102B** (1981), 27-31.

O.M. Khudaverdian, A.P. Nersessian.
On Geometry of Batalin-Vilkovisky Formalism. Mod. Phys. Lett. A, v. 8 (1993), No. 25, p. 2377-2385.
Batalin-Vilkovisky Formalism and Integration Theory on Manifolds. J. Math. Phys., v. 37 (1996), p. 3713-3724.

A.S. Schwarz. *Geometry of Batalin Vilkovsiky formalism*. Comm.Math.Phys., **155**, (1993), no.2, 249–260.

H.M. Khudaverdian.

Geometry of Superspace with Even and Odd Brackets. J. Math. Phys. v. 32 (1991) p. 1934-1937 (Preprint of the Geneva University, UGVA–DPT 1989/05–613).

Odd Invariant Semidensity and Divergence-like Operators on Odd Symplectic Superspace. Comm. Math. Phys., v. 198 (1998), p. 591-606.

Semidensities on odd symplectic supermanifold., Comm. Math. Phys., v. 247 (2004), pp. 353-390 (Preprint of Max-Planck-Institut für Mathematik, MPI-135 (1999), Bonn.)

H.M. Khudaverdian, T.Voronov

On Odd Laplace operators. Lett. Math. Phys. 62 (2002), 127-142

On odd Laplace operators. II. In *Amer. Math. Soc. Transl.* (2), Vol. 212, 2004, pp.179–205

Differential forms and odd symplectic geometry. Amer. Math. Soc. Transl (2) Vol 224, 2008 pp.159—171

First order operator (on functions)

$$L = T^a(x)\partial_a + R(x)\,, \qquad \left(\partial_a \leftrightarrow \frac{\partial}{\partial x^a}\right)$$

Change of coordinates $x^a = x^a(x^{a'})$

$$\partial_a = x_a^{a'}\partial_{a'}, \qquad \left(x_a^{a'} = \frac{\partial x^{a'}}{\partial x^a}\right)$$

$$L = T^a(x)\partial_a + R(x) = \underbrace{T^a(x)x_a^{a'}}_{T^{a'}}\partial_{a'} + R(x)$$

$$L = \underbrace{T^a(x)\partial_a}_{\text{vector field}} + \underbrace{R(x)}_{\text{scalar}}$$

Second order operator (on functions)

$$\Delta = \frac{1}{2} S^{ab}(x)\partial_a\partial_b + T^a(x)\partial_a + R(x)\,,$$

Change of coordinates $x^a = x^a(x^{a'})$

$$\Delta = \frac{1}{2} S^{ab}(x)\partial_a\partial_b + \dots = \frac{1}{2}\underbrace{x_a^{a'} S^{ab} x_b^{b'}}_{S^{a'b'}}\partial_{a'}\partial_{b'} + \dots$$

S^{ab} defines symmetric contravariant tensor of rank 2 on M.

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Change of coordinates $x^a = x^a(x^{a'})$

$$\Delta = \frac{1}{2} S^{ab}(x)\partial_a\partial_b + \dots = \frac{1}{2} \underbrace{x_a^{a'} S^{ab} x_b^{b'}}_{S^{a'b'}} \partial_{a'}\partial_{b'} + \dots$$

S^{ab} defines symmetric contravariant tensor of rank 2 on M.

Quadratic polynomial $H_\Delta = \frac{1}{2} S^{ab} p_a p_b$ on T^*M is the principal symbol of the operator $\Delta = \frac{1}{2}\partial_a S^{ab}(x)\partial_b + \dots$. (Linear polynomial $H_L = T^a p_a$ on T^*M is the principal symbol of the operator $L = T^a\partial_a + \dots$.)

$$\Delta = \frac{1}{2}\underbrace{S^{ab}(x)\partial_a\partial_b}_{\text{top component}} + T^a(x)\partial_a + R(x) \tag{1}$$

The top component
symmetric tensor field $S^{ab}\partial_a \otimes \partial_b$ on M.

If $S \equiv 0$ then Δ becomes first order operator
$T^a\partial_a$ is vector field.

What about the geometrical meaning of the term $T^a\partial_a$ if $S \neq 0$?

$$\Delta = \frac{1}{2} \underbrace{S^{ab}(x)\partial_a\partial_b}_{\text{top component}} + T^a(x)\partial_a + R(x) \tag{1}$$

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What about the geometrical meaning of the term $T^a\partial_a$ if $S \neq 0$?
To study this question consider the difference

$$\Delta^+ - \Delta\,,$$

where Δ^+ is defined via a scalar product

$$\langle \Delta f, g \rangle = \langle f, \Delta^+ g \rangle\,.$$

Scalar product and volume form

$$\langle f, g\rangle_\rho = \int_M f(x)g(x)\rho(x)Dx, \qquad \rho(x)Dx \text{ is a volume form}.$$

$$\rho(x)Dx = \rho(x(x')) \left|\frac{Dx}{Dx'}\right| Dx' = \rho(x(x')) \det\left(\frac{\partial x^a}{\partial x^{a'}}\right) Dx'$$

$$\langle \Delta f, g\rangle_\rho = \int_M \Delta(f(x))g(x)\rho(x)Dx$$

$$= \int_M f(x)\Delta^+(g(x))\rho(x)Dx = \langle f, \Delta^+ g\rangle_\rho .$$

By integrating by parts we have

$$\langle \Delta f, g\rangle_\rho = \int_M \underbrace{\left(\frac{1}{2}S^{ab}(x)\partial_a\partial_b f + T^a(x)\partial_a f + R(x)f\right)}_{\Delta f} g(x)\rho(x)Dx =$$

$$\int_M f(x) \underbrace{\left(\frac{1}{2\rho}\partial_a\left(\partial_b\left(S^{ab}\rho g\right)\right) - \frac{1}{\rho}\partial_a\left(T^a\rho g\right) + Rg\right)}_{\Delta^+ g} \rho(x)Dx = \langle f, \Delta^+ g\rangle_\rho$$

$$\Delta^+ - \Delta = \underbrace{\left(\partial_b S^{ab} - 2T^a + S^{ab}\partial_b \log\rho\right)\partial_a}_{\text{vector field}} + \dots$$

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$$\langle \Delta f, g\rangle_\rho = \int_M \underbrace{\left(\frac{1}{2}S^{ab}(x)\partial_a\partial_b f + T^a(x)\partial_a f + R(x)f\right)}_{\Delta f} g(x)\rho(x)Dx =$$

$$\int_M f(x)\underbrace{\left(\frac{1}{2\rho}\partial_a\left(\partial_b\left(S^{ab}\rho g\right)\right) - \frac{1}{\rho}\partial_a\left(T^a\rho g\right) + Rg\right)}_{\Delta^+ g}\rho(x)Dx = \langle f, \Delta^+ g\rangle_\rho$$

$$\Delta^+ - \Delta = \underbrace{\left(\partial_b S^{ab} - 2T^a + S^{ab}\partial_b \log\rho\right)\partial_a}_{\text{vector field}} + \ldots$$

Claim: for an operator $\Delta = S^{ab}\partial_a\partial_b + T^a\partial_a + R$, the expression $\gamma^a = \partial_b S^{ab} - 2T^a$, is an (upper) connection on volume forms.

Connection on volume forms

Connection ∇ on volume forms defines covariant derivative

$$\nabla_a(\rho Dx) = (\partial_a + \gamma_a)\rho(x)Dx\,, \qquad \gamma_a Dx = \nabla_a(Dx)\,.$$

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Transformation of the symbol $\gamma_a(x)$: $x^a = x^a(x^{a'})$

$$\gamma_a Dx = x_a^{a'}\nabla_{a'}\left(\det\frac{\partial x}{\partial x'}Dx'\right) = x_a^{a'}\left(\partial_{a'}\left(\log\det\frac{\partial x}{\partial x'}\right) + \gamma_{a'}\right)Dx$$

$$\gamma_a = x_a^{a'}\gamma_{a'} - x_{b'}^{b}x_{ba}^{b'}\,.$$

The difference of two connections is a vector field:

$$\gamma_a' - \gamma_a = \text{vector field}$$

Examples of connections

Example (Connection defined by a chosen volume form)

Let $\rho_{(0)}(x)Dx$ be a non-vanishing volume form. Define:

$$\nabla_a^{(0)}(\rho(x)Dx) = \partial_a\left(\frac{\rho Dx}{\rho_{(0)}Dx}\right)\rho_{(0)}Dx = (\partial_a\rho(x) - \partial_a \log\rho_{(0)}(x))\rho_{(0)}Dx\,,$$

$$\gamma_a^{(0)} = -\partial_a \log\rho_{(0)}(x)$$

$$\nabla_a^{(0)}\left(\rho_{(0)}(x)Dx\right) \equiv 0\,.$$

It is the connection induced by a volume form

Example (Connection defined by a Riemannian structure)

Let M be Riemannian manifold with metric $g_{ab}dx^a dx^b$.

$$\rho_{(g)}(x)Dx = \sqrt{\det g}Dx\,, \qquad \text{(canonical volume form)}$$

$$\gamma_a^{(g)} = -\partial_a \log \rho_{(g)}(x) = -\frac{1}{2}\partial_a \log \det g = -\frac{1}{2} g^{bc}\partial_a g_{bc} = -\Gamma^b_{ab}\,,$$

where Γ^a_{bc} are the Christoffel symbols of the Levi-Civita connection.

Here connection on volume forms is the trace of Christoffel symbols.

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where Γ^a_{bc} are the Christoffel symbols of the Levi-Civita connection.

Here connection on volume forms is the trace of Christoffel symbols.

Volume form connection $\gamma_a = -\partial_a \log \rho$ is a flat connection: its curvature vanishes

$$f_{ab} = \partial_a \gamma_b - \partial_b \gamma_a = -\partial_a \partial_b \log \rho + \partial_b \partial_a \log \rho = 0.$$

Connection on volume forms and divergence

If ∇ is a connection on volume forms $\nabla_a \rho Dx = (\partial_a + \gamma_a)\rho(x)Dx$ then one can define a divergence operator on vector fields:

$$\text{For} \qquad \mathbf{X} = X^a \partial_a, \ \mathrm{div}_\gamma \mathbf{X} = \partial_a X^a - \gamma_a X^a .$$

If the connection ∇ is induced by a volume form $\rho(x)Dx$

$$\gamma_a^{(\rho)} = -\partial_a \log \rho_a, \qquad \text{then}$$

$$\mathrm{div}_{\gamma^{(\rho)}} \mathbf{X} = \partial_a X^a - \gamma_a^{(\rho)} X^a = \partial_a X^a + X^a \partial_a \log \rho(x) = \frac{\mathscr{L}_\mathbf{X} \rho Dx}{\rho Dx} .$$

If $\rho(x)Dx = \sqrt{\det g} Dx$ is the canonical volume form on a Riemannian manifold, then $\mathrm{div}\,\mathbf{X} = (\partial_b + \Gamma^a_{ab})X^b$.

Returning to operators

For operator $\Delta = \frac{1}{2} S^{ab}\partial_a\partial_b + T^a\partial_a + R$ we consider adjoint operator Δ^+ with respect to the scalar product induced by a volume form $\rho(x)Dx$.

$$\Delta^+ - \Delta = \underbrace{\left(\partial_b S^{ab} - 2T^a + S^{ab}\partial_b \log\rho\right)}_{\text{vector field}}\partial_a + \dots$$

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$$\Delta^+ - \Delta = \underbrace{\left(\partial_b S^{ab} - 2T^a + S^{ab}\partial_b \log\rho\right)}_{\text{vector field}} \partial_a + \dots$$

Consider the flat connection $\gamma_a^{(\rho)} = -\partial_a \log\rho$ induced by a volume form $\rho(x)Dx$. We obtain

$$\partial_b S^{ab} - 2T^a = \text{vector field} - S^{ab}\partial_b \log\rho = \text{vector field} + S^{ab}\gamma_b^{(\rho)}$$

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$$\partial_b S^{ab} - 2T^a = \text{vector field} - S^{ab}\partial_b \log\rho = \text{vector field} + S^{ab}\gamma_b^{(\rho)}$$

$$\partial_b S^{ab} - 2T^a = \text{vector field} + \gamma^{a(\rho)} = \gamma^a \ \text{upper connection}$$

Geometry of second order operator (on functions)

$$\partial_b S^{ab} - 2T^a = \gamma^a \text{ upper connection on volume forms}$$

$$\Delta = \frac{1}{2} S^{ab}\partial_a\partial_b f + T^a\partial_a + R = \frac{1}{2} S^{ab}\partial_a\partial_b + \frac{1}{2}\left(\partial_b S^{ba} - \gamma^a\right)\partial_a + R\,,$$

$$\Delta f = \frac{1}{2}\partial_a\left(\underbrace{S^{ab}}_{\text{tensor}}\partial_b f\right) - \frac{1}{2}\underbrace{\gamma^a}_{\text{connection}}\partial_a f + \underbrace{R}_{\text{scalar}} f\,,$$

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Upper connection γ^a on volume forms defines contravariant derivative:

$$\nabla^a\left(\rho(x)Dx\right) = \left(S^{ab}\partial_b + \gamma^a\right)\rho Dx\,.$$

If γ_a is connection on volume form then $\gamma^a = S^{ab}\gamma_b$ is upper connection.

Example: Laplace-Beltrami operator

Fix a volume form $\rho(x)Dx$ and consider the induced flat connection $\gamma_a = -\partial_a \log \rho$. Fix scalar $R = 0$. Then

$$\Delta = \frac{1}{2}\partial_a\left(S^{ab}\partial_b\right) - \frac{1}{2}\gamma^a\partial_a + R =$$

$$\frac{1}{2}\partial_a\left(S^{ab}\partial_b\right) + \frac{1}{2}S^{ab}\partial_b \log \rho\, \partial_a = \frac{1}{2}\frac{1}{\rho}\partial_a\left(\rho S^{ab}\partial_b\right) .$$

In the Riemannian case $S^{ab} = g^{ab}$ and $\rho(x) = \sqrt{\det g}$.

Algebra of densities

Under a change of coordinates a density of weight σ is multiplied by the σ-th power of the Jacobian of the coordinate transformation:

$$s(x)|Dx|^{\sigma} = s(x(x')) \left|\frac{Dx}{Dx'}\right|^{\sigma} |Dx'|^{\sigma} = s(x(x')) \left(\det\left(\frac{\partial x}{\partial x'}\right)\right)^{\sigma} |Dx'|^{\sigma}.$$

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Density of weight $\sigma = 0$ is a usual scalar function.
Density of weight $\sigma = 1$ is a volume form.
Wave function Ψ is a density of weight $\sigma = \frac{1}{2}$ (semi-density).

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Density of weight $\sigma = 0$ is a usual scalar function.
Density of weight $\sigma = 1$ is a volume form.
Wave function Ψ is a density of weight $\sigma = \frac{1}{2}$ (semi-density).
Product of two densities:

$$s_1(x)|Dx|^{\sigma_1} \cdot s_2(x)|Dx|^{\sigma_2} = s'(x)|Dx|^{\sigma_1+\sigma_2} .$$

Canonical scalar product of densities

Definition

$$\langle s_1(x)|Dx|^{\sigma_1}, s_2(x)|Dx|^{\sigma_2}\rangle = \int_M s_1(x)s_2(x)Dx, \qquad \text{if } \sigma_1+\sigma_2 = 1,$$

$$\langle s_1(x)|Dx|^{\sigma_1}, s_2(x)|Dx|^{\sigma_2}\rangle = 0 \qquad \text{if } \sigma_1+\sigma_2 \neq 1$$

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$$\langle s_1(x)|Dx|^{\sigma_1}, s_2(x)|Dx|^{\sigma_2}\rangle = 0 \qquad \text{if } \sigma_1+\sigma_2 \neq 1$$

Symbolic notation:
$s(x)|Dx|^{\sigma} \leftrightarrow s(x)t^{\sigma}$. Density $a(x,t) = \sum a_k t^{\sigma_k}$

$$\langle a(x,t), b(x,t)\rangle = \int_M \mathrm{Res}\left(\frac{a(x,t)b(x,t)}{t^2}\right) Dx\,.$$

Differential operators on densities

Differential operators $D = D(x, t, \frac{\partial}{\partial x}, \frac{d}{dt})$ act on densities $a(x,t) = \sum a_k(x) t^{\sigma_k}$, $(t^\sigma \leftrightarrow |Dx|^\sigma)$.

Examples

Weight operator: $\hat{\sigma} = t\frac{d}{dt}$. $t\frac{d}{dt}(a(x)t^\sigma) = \sigma a(x) t^\sigma$.

Lie derivative:

$$\mathscr{L}_{\mathbf{X}} = X^a \frac{\partial}{\partial x^a} + \frac{\partial X^a}{\partial x^a} t \frac{d}{dt}$$

$$\mathscr{L}_{\mathbf{X}}(a(x)|Dx|^\sigma) = \left(X^a \frac{\partial a(x)}{\partial x^a} + \sigma \frac{\partial X^a}{\partial x^a} a(x)\right)|Dx|^\sigma, \ .$$

Examples of adjoints

$\partial_a^+ = -\partial_a, t^+ = t, \left(\frac{d}{dt}\right)^+ = -\frac{d}{dt} + \frac{2}{t}, \hat{\sigma}^+ = 1 - \hat{\sigma}$.

Second order operator on the density algebra

Contravariant tensor S^{ab}, upper connection γ^a	$\longleftrightarrow$	Second order self-adjoint operator on algebra of densities

(H.Kh., T.Voronov 2003)

$$\Delta a(x,t) = \Delta^+ a(x,t) =$$

$$\frac{1}{2}\left(\partial_a S^{ab}\partial_b + (2\hat{\sigma}-1)\gamma^a\partial_a + \hat{\sigma}\partial_a\gamma^a + \hat{\sigma}(\hat{\sigma}-1)\theta\right) a(x,t)\,.$$

Here $\theta = \gamma^a S_{ab}\gamma^b = \gamma^a\gamma_a$ (in the case if S^{ab} is invertible).

Second order operator on the density algebra

Contravariant tensor S^{ab}, upper connection γ^a	$\longleftrightarrow$	Second order self-adjoint operator on algebra of densities

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Here $\theta = \gamma^a S_{ab}\gamma^b = \gamma^a\gamma_a$ (in the case if S^{ab} is invertible).

In the general case θ is an object such that for an arbitrary connection γ'_a $\theta - \gamma'_a S^{ab}\gamma'_b - 2\partial_a(\gamma^a - S^{ab}\gamma'_b)$ is a scalar. It is a Brans-Dicke type "scalar".

Canonical pencil of operators

Restricting the operator Δ on densities of weight σ we arrive at the operator pencil Δ_σ,

$$\Delta_\sigma\left(a(x)|Dx|^\sigma\right) =$$

$$\frac{1}{2}\left(\partial_a S^{ab}\partial_b + (2\sigma-1)\gamma^a\partial_a + \sigma\partial_a\gamma^a + \sigma(\sigma-1)\theta\right)a(x)|Dx|^\sigma\,,$$

$\sigma \in \mathbf{R}$.

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$\sigma \in \mathbf{R}$.

Theorem ("Universality" property)

Let L be an arbitrary second order operator acting on densities of the weight σ. If $\sigma \neq 0, \frac{1}{2}, 1$ then there exists a unique canonical pencil which passes through the operator L, $L = \Delta_\sigma$. (H.Kh., T.Voronov)

Special case: operators on semidensities, $\sigma = \frac{1}{2}$.

Fix S^{ab}. Choose an arbitrary connection γ_a. Consider the canonical pencil at $\sigma = \frac{1}{2}$.

$$\Delta^{\gamma}_{\frac{1}{2}}\left(a(x)\sqrt{|Dx|}\right) = \frac{1}{2}\left(\partial_a\left(S^{ab}\partial_b a(x)\right) + \frac{\partial_a\gamma^a}{2}a(x) - \frac{\gamma^a\gamma_a}{4}a(x)\right)\sqrt{|Dx}$$

How this operator changes if we change the connection γ?

$$\gamma \to \gamma' = \gamma + \mathbf{X}, \quad \Delta^{\gamma}_{\frac{1}{2}} \to \Delta^{\gamma'}_{\frac{1}{2}} = \Delta^{\gamma}_{\frac{1}{2}} + \frac{1}{4}\partial_a X^a - \frac{1}{8}\left(2\gamma_a X^a + X_a X^a\right) =$$

$$\Delta^{\gamma}_{\frac{1}{2}} + \frac{1}{4}\left(\partial_a X^a - \gamma_a X^a\right) - \frac{1}{8}\mathbf{X}^2 = \Delta^{\gamma}_{\frac{1}{2}} + \frac{1}{4}\left(\mathrm{div}_{\gamma}\mathbf{X} - \frac{1}{2}\mathbf{X}^2\right).$$

$$\Delta^{\gamma}_{\frac{1}{2}} = \Delta^{\gamma'}_{\frac{1}{2}} \quad \Leftrightarrow \quad \mathrm{div}_{\gamma}\mathbf{X} - \frac{1}{2}\mathbf{X}^2 = 0.$$

Groupoid of connections

Let A be an affine space of all connections on volume forms.

Arrow: $\gamma \xrightarrow{\mathbf{X}} \gamma'$ such that $\gamma, \gamma' \in A$ and $\gamma' = \gamma + \mathbf{X}$.

Set S of admissible arrows: $S = \{\gamma \xrightarrow{\mathbf{X}} \gamma' : \quad \mathrm{div}_\gamma \mathbf{X} - \frac{1}{2}\mathbf{X}^2 = 0\}$

Inverse arrow: If $\gamma \xrightarrow{\mathbf{X}} \gamma' \in S$ then $\gamma' \xrightarrow{-\mathbf{X}} \gamma \in S$.

(If $\mathrm{div}_\gamma \mathbf{X} - \frac{1}{2}\mathbf{X}^2 = 0$ then $-\mathrm{div}_{\gamma+\mathbf{X}} \mathbf{X} - \frac{1}{2}\mathbf{X}^2 = 0$).

Multiplication of arrows: if $\gamma_1 \xrightarrow{\mathbf{X}} \gamma_2, \ \gamma_2 \xrightarrow{\mathbf{Y}} \gamma_3 \in S$ then $\gamma_1 \xrightarrow{\mathbf{X}+\mathbf{Y}} \gamma_3 \in S$.

$$(\text{if } \mathrm{div}_{\gamma_1} \mathbf{X} - \frac{1}{2}\mathbf{X}^2 = \mathrm{div}_{\gamma_2} \mathbf{Y} - \frac{1}{2}\mathbf{Y}^2 = 0 \text{ then } \mathrm{div}_{\gamma_1} (\mathbf{X}+\mathbf{Y}) - \frac{1}{2}(\mathbf{X}+\mathbf{Y})^2 = 0\,.)$$

We call this groupoid the Batalin-Vilkovisky groupoid.
(H.Kh., T. Voronov.)

Conclusion

Operator $\Delta^{\gamma}_{\frac{1}{2}}$ depends not on a connection but only on its equivalence class, the groupoid orbit $\mathcal{O}_\gamma$ of a connection γ,

$$\mathcal{O}_\gamma = \{\gamma' : \quad \gamma \xrightarrow{\mathbf{X}} \gamma' \in S\}.$$

$$\Delta^{\gamma}_{\frac{1}{2}} = \Delta^{\gamma'}_{\frac{1}{2}} \qquad \Leftrightarrow \qquad \mathrm{div}_{\gamma}\mathbf{X} - \frac{1}{2}\mathbf{X}^2 = 0\,.$$

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$$\mathcal{O}_\gamma = \{\gamma' : \quad \gamma \xrightarrow{\mathbf{X}} \gamma' \in S\}.$$

$$\Delta^{\gamma}_{\frac{1}{2}} = \Delta^{\gamma'}_{\frac{1}{2}} \qquad \Leftrightarrow \qquad \mathrm{div}_{\gamma}\mathbf{X} - \frac{1}{2}\mathbf{X}^2 = 0\,.$$

Where such operators naturally arise?

Consider a supermanifold M with coordinates $z^A = \{\underbrace{x^a}_{\text{even}}, \underbrace{\theta^\alpha}_{\text{odd}}\}$. Let S^{AB} be a (super)symmetric contravariant tensor on M:

$$S^{AB} = S^{BA}(-1)^{p(A)p(B)}\,.$$

It defines $\Delta = S^{AB}\partial_A\partial_B + \dots$.

Suppose S^{AB} is invertible.

1-st case . S^{AB} is an even tensor: $p(S^{AB}) = p(A) + p(B)$.

$S^{AB} = g^{AB}$ defines an even Riemannian structure.

There exists the canonical volume form and the canonical flat connection on volume forms:

$$\rho(z)|Dz| = \sqrt{\mathrm{Ber}\, g_{AB}},\ \gamma_A = -\partial_A \log\rho(z)\,.$$

Moreover there exists the unique Levi-Civita connection Γ^A_{BC} and

$$\gamma_A = -\partial_A \log\rho(z)|Dz| = -(-1)^B\Gamma^B_{BA}\,.$$

2-nd case . S^{AB} is an odd tensor: $p(S^{AB}) = 1 + p(A) + p(B)$.
$S^{AB} = \Omega^{AB}$ defines an odd symplectic structure [1]:
$\{z^A, z^B\} = (-1)^A \Omega^{AB}$.

There are no canonical volume form (no Liouville Theorem!) and no canonical flat connection on volume forms.
There are many affine connections compatible with the symplectic structure. One cannot choose a unique "Levi-Civita" connection Γ^A_{BC}.

One cannot choose a distinguished connection on volume forms.

Can we choose a class of connections?

[1] We need to impose the additional condition $(\Omega^{AB}\pi_A\pi_B, \Omega^{AB}\pi_A\pi_B) = 0$ where $(,)$ is a canonical Poisson bracket on the cotangent bundle T^*M, providing the Jacobi identity for the odd bracket $\{f,g\} = (f, (\Omega^{AB}\pi_A\pi_B, g))$.

Canonical class of connections

Definition

We say that γ_A is a Darboux flat connection if there exist Darboux coordinates such that $\gamma_A \equiv 0$ in these Darboux coordinates.

Theorem

All Darboux flat connections belong to the same orbit of the Batalin-Vilkovisky groupoid. That means that for two Darboux flat connections γ_1, γ_2

$$\gamma_1 \xrightarrow{\mathbf{X}} \gamma_2 \in S, \text{ i.e. } \operatorname{div} \mathbf{X} - \frac{1}{2}\mathbf{X}^2 = 0\,,$$

(I.A.Batalin, G.A.Vilkovisky [2]*—H.Kh.—H.Kh.,T.Voronov)*

[2]The statement relies on the Batalin-Vilkovisky identity: $\Omega^{AB}\partial_A\partial_B\sqrt{\operatorname{Ber}\left(\frac{\partial z^A}{\partial z^{A'}}\right)} = 0$ for Darboux coordinates $z^A, z^{A'}$.

Example. Canonical Δ-operator on semidensitites

Let γ be an arbitrary Darboux flat connection and $\{z^A\}$ be arbitrary Darboux coordinates. Then

$$\Delta^{\mathcal{O}_\gamma}_{\frac{1}{2}}\left(a(z)\sqrt{|Dz|}\right) =$$

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$$\frac{1}{2}\left(\partial_A\left(\Omega^{AB}\partial_B a(z)\right) + \frac{\partial_A\gamma^A}{2}a(z) - \frac{\gamma^A\gamma_A}{4}a(z)\right)\sqrt{|Dz|}$$

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$$= \frac{1}{2}\Omega^{BA}\partial_A\partial_B a(z)\sqrt{|Dz|},$$

since Ω^{BA} is a constant tensor in Darboux coordinates and according to Theorem above, $\frac{\partial_A\gamma^A}{2} - \frac{\gamma^A\gamma_A}{4} = 0$ for an arbitrary Darboux flat connection.

Analogue of mean curvature for an odd symplectic structure.

Let M be an odd symplectic supermanifold equipped with a volume form $\rho(z)|Dz|$.
Let C be a surface of codimension $(1|1)$ in M and $\Psi(z)$ be an odd vector field which is symplectoorthogonal to the surface M.
Consider

$$A(\nabla,\Psi) = \mathrm{Tr}\left(\Pi\left(\nabla\Psi\right)\right) - \mathrm{div}_\rho \Psi\,,$$

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In the even Riemannian case (surface of codimension $(1|0)$) one can take the canonical Levi-Civita connection ∇_{LC} and the Riemannian volume form. Then

$$A(\nabla_{LC}, \Psi) = |\Psi| \cdot \text{mean curvature of the surface } C \,.$$

In the odd symplectic case there is no preferred affine connection compatible with the symplectic structure.
Consider the class of Darboux flat affine connections.
(Connection is Darboux flat if there exist Darboux coordinates such that Christoffel symbols $\Gamma^A_B C \equiv 0$ in these Darboux coordinates)

Theorem

The magnitude $A(\nabla,\Psi)$ does not depend on a connection in the class of Darboux flat connections:

$$A(\nabla,\Psi) = A(\nabla',\Psi)$$

for two arbitrary Darboux flat connections ∇ and ∇'.

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This construction reveals the geometrical meaning of odd invariant semidensity obtained in 1984 (H.Kh., R.Mkrtchyan).

References

O.M. Khudaverdian, A.S. Schwarz, Yu.S. Tyupkin. *Integral invariants for Supercanonical Transformations.* Lett. Math. Phys., v. 5 (1981), p. 517-522.

References

O.M. Khudaverdian, A.S. Schwarz, Yu.S. Tyupkin. *Integral invariants for Supercanonical Transformations.* Lett. Math. Phys., v. 5 (1981), p. 517-522.

O.M. Khudaverdian, R.L. Mkrtchian. *Integral Invariants of Buttin Bracket.* Lett. Math. Phys. v. 18 (1989), p. 229-231 (Preprint EFI–918–69–86- Yerevan (1986)).

I.A.Batalin, G.A Vilkovisky. *Gauge algebra and quantization.* Phys.Lett., **102B** (1981), 27-31.

O.M. Khudaverdian, A.P. Nersessian.
On Geometry of Batalin-Vilkovisky Formalism. Mod. Phys. Lett. A, v. 8 (1993), No. 25, p. 2377-2385.
Batalin-Vilkovisky Formalism and Integration Theory on Manifolds. J. Math. Phys., v. 37 (1996), p. 3713-3724.

A.S. Schwarz. *Geometry of Batalin Vilkovsiky formalism*. Comm.Math.Phys., **155**, (1993), no.2, 249–260.

H.M. Khudaverdian.
Geometry of Superspace with Even and Odd Brackets. J. Math. Phys. v. 32 (1991) p. 1934-1937 (Preprint of the Geneva University, UGVA–DPT 1989/05–613).
Odd Invariant Semidensity and Divergence-like Operators on Odd Symplectic Superspace. Comm. Math. Phys., v. 198 (1998), p. 591-606.
Semidensities on odd symplectic supermanifold., Comm. Math. Phys., v. 247 (2004), pp. 353-390 (Preprint of Max-Planck-Institut für Mathematik, MPI-135 (1999), Bonn.)
H.M. Khudaverdian, T.Voronov
On Odd Laplace operators. Lett. Math. Phys. 62 (2002), 127-142
On odd Laplace operators. II. In *Amer. Math. Soc. Transl.* (2), Vol. 212, 2004, pp.179–205
Differential forms and odd symplectic geometry. Amer. Math. Soc. Transl (2) Vol 224, 2008 pp.159—171