Kaluza-Klein theory revisited: projective structures and differential operators on algebra of densities.

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Contents

In mathematical physics it is very useful to consider differential operators acting on densities of various weights on a manifold M. To study the geometry of such operators one can consider an operator pencil Δ_t where for an arbitrary real t an operator Δ_t acts on densities of weight t defined on a manifold M. Pencils of this kind can be interpreted as differential operators on a certain algebra of functions on extended manifold $\hat{M}$. For second order operators the study of their geometry naturally fits into a Kaluza-Klein framework. For such an operator the related geometry is defined by principal symbol (”metric on M”), a connection on volume forms (”gauge field”) and a function related with the scalar term (”Brans-Dicke scalar”). This becomes useful to study important and beautiful geometrical properties of second order differential operators. The extended manifold $\hat{M}$ can be identified with Thomas bundle dating back in projective geometry to 1920. We see that study of the extended manifold $\hat{M}$ provides constructions on intersection of classical differential geometry and gravitational theory. Such

Abstract

Operator pencil $\{\Delta_\lambda\}$, $\lambda \in \mathbf{R}$
(An operator Δ_λ acts on densities of weight λ on manifold M)

Operator pencil $=$ differential operators on a certain algebra of functions on extended manifold $\hat{M}$.

For second order operator

Principal symbol— "metric on M",

A connection on volume forms— "gauge field"

A function related with the scalar term — "Brans-Dicke scalar"

Abstract...

The extended manifold $\hat{M}$ can be identified with Thomas bundle dating back in projective geometry to 1920.
Study of the extended manifold $\hat{M}$ provides constructions on intersection of classical differential geometry and gravitational theory.
Such investigations can be traced to H.Weil, Veblen, T.Y.Thomas, Pauli and Jordan.

Densities of weight λ on manifold M

$\mathbf{s}(x) = s(x)|Dx|^\lambda$ is a density of weight λ ($\lambda \in \mathbf{R}$). Under changing of lcoal coordinates $x = x(x')$

$$s(x)|Dx|^\lambda = s(x')|Dx'|^\lambda, \quad \text{i.e.}$$

$$s(x) = s\left(x\left(x'\right)\right) \det\left(\frac{\partial x}{\partial x'}\right)^\lambda .$$

Functions— densities of weight $\lambda = 0$
Volume forms— densities of weight $\lambda = 1$
Wave-functions— densities of weight $\lambda = 1/2$

$\mathcal{F}_\lambda = \mathcal{F}_\lambda(\mathcal{M})$—space of densities of weight λ on manifold M

M is orientable manifold with fixed orientation, i.e. all transition functions $x' = x'(x)$ have positive determinants.

Operators on space $\mathcal{F}_\lambda(M)$ of densities of weight λ. Operator pencils

Operator pencil $\{\Delta_\lambda\}$.

$\forall\lambda \in \mathbf{R}$, Δ_λ is an operator acting on the space $\mathcal{F}_\lambda(M)$ of densities of weight λ.

(We suppose that dependence on λ is polynomial)

Example

$$\{\Delta_\lambda\}\colon \Delta_\lambda = \frac{d^2}{dx^2} + (2\lambda+1)\frac{d}{dx} + (1-\lambda^2)\cos x\,,$$

$\lambda = 0$, $\Delta_0 = \frac{d^2}{dx^2} + \frac{d}{dx} + \cos x$,

$\lambda = 1/2$, $\frac{d^2}{dx^2} + 2\frac{d}{dx} + 3/4\cos x$,,

$\lambda = 1$, $\frac{d^2}{dx^2} + 3\frac{d}{dx}$

Algebra of densities.

We incorporate all the densities of various weights λ in an algebra $\mathscr{F}(M) = \oplus_\lambda \mathscr{F}_\lambda(M)$.

If $\mathbf{s}_1(x) = s_1(x)|Dx|^{\lambda_1}$ and $\mathbf{s}_2(x) = s_2(x)|Dx|^{\lambda_2}$ are densities of weights λ_1, λ_2 respectively, then their product,

$$\mathbf{s} = \mathbf{s}_1 \cdot \mathbf{s}_2 = s_1(x)|Dx|^{\lambda_1} s_2(x)|Dx|^{\lambda_2} = s_1(x)s_2(x)|Dx|^{\lambda_1+\lambda_2}$$

is a density of the weight $\lambda = \lambda_1 + \lambda_2$.

Local coordinates for algebra of densities

$$\mathscr{F}(M) \ni \mathbf{s}(x) = s_1(x)|Dx|^{\lambda_1} + s_2(x)|Dx|^{\lambda_2} + \cdots + s_k(x)|Dx|^{\lambda_k} ,$$

$$|Dx| \sim t ,$$

$$\mathscr{F}(M) \ni \mathbf{s}(x) = s_1(x)t^{\lambda_1} + s_2(x)t^{\lambda_2} + \cdots + s_k(x)t^{\lambda_k} ,$$

$$(x^\mu, t) \mapsto (x^{\mu'}, t'), \quad x^{\mu'} = x^{\mu'}(x^\mu),\, t'(x^\mu, t) = \left(\det\left(\frac{\partial x^{\mu'}}{\partial x^\mu}\right)\right) t .$$

x^μ are local coordinates on the manifold $\hat{M}$.
(x^μ, t) are local coordinates on the extended manifold $\hat{M}$
The extended manifold $\hat{M}$ has one extra dimension.
$\begin{matrix}\hat{M}\\ \downarrow\\ M\end{matrix}$ is the frame bundle of determinant bundle $\det TM$, (TM is a tangent bundle). The fibre is 1-dimensional.

Weight operator $\hat{w}$ on algebra of densities

Definition

$\hat{w}$ is linear operator such that

$$\hat{w}(\mathbf{s}) = \lambda \mathbf{s}\,,$$

If $\mathbf{s}$ is a density of weight λ, $\mathbf{s}(x) = s(x)|Dx|^{\lambda}$ ($\mathbf{s}(x) = s(x)t^{\lambda}$).

Let $\mathbf{s}_1$ be a density of weight λ_1 and $\mathbf{s}_2$ be a density of weight λ_2

$$\text{then } \hat{w}(\mathbf{s}_1 \cdot \mathbf{s}_2) = (\lambda_1 + \lambda_2)\mathbf{s}_1 \cdot \mathbf{s}_2 = \hat{w}\left(\mathbf{s}_1\right) \cdot \mathbf{s}_2 + \mathbf{s}_1 \cdot \hat{w}\left(\mathbf{s}_2\right).$$

Leibnitz rule (it is as $\frac{\partial}{\partial x}(fg) = \frac{\partial f}{\partial x} g + f\frac{\partial g}{\partial x}$.)

$\hat{w}$ is first order differential operator on algebra of densities

In local coordinates $\hat{w} = t\frac{\partial}{\partial t}$:

$$\hat{w}\left(s(x)|Dx|^{\lambda}\right) = \hat{w}\left(s(x)t^{\lambda}\right) = t\frac{\partial}{\partial t}\left(s(x)t^{\lambda}\right) = \lambda s(x)t^{\lambda} = \lambda s(x)|Dx|^{\lambda}.$$

Operator pencils $\leftrightarrow$ Operators on algebra of densities

$$\{\Delta_\lambda\} \mapsto \hat{\Delta} \colon \hat{\Delta}\big|_{\hat{w}=\lambda} = \Delta_\lambda$$

Example

$$\{\Delta_\lambda\} \colon \Delta_\lambda = \frac{d^2}{dx^2} + (2\lambda + 1)\frac{d}{dx} + (1 - \lambda^2)\cos x$$

$$\updownarrow$$

$$\hat{\Delta} = \frac{\partial^2}{\partial x^2} + (2\hat{w} + 1)\frac{\partial}{\partial x} + (1 - \hat{w}^2)\cos x$$

Another example

$$\{\Delta_\lambda\}\colon\ \Delta_\lambda = A(\lambda)S^{\mu\nu}\partial_\mu\partial_\nu + B(\lambda)T^\mu\partial_\mu + C(\lambda)R,$$

$$\updownarrow$$

$$\hat{\Delta} = A(\hat{w})S^{\mu\nu}\partial_\mu\partial_\nu + B(\hat{w})T^\mu\partial_\mu + C(\hat{w})R\,.$$

Here $A(\lambda), B(\lambda)$ and $C(\lambda)$ are polynomials on λ.
E.g. if $A(\lambda) = 1+\lambda$, $B(\lambda) = \lambda^2$, $C(\lambda) = 1$ then an operator pencil

$$\{\Delta_\lambda\}\colon \quad \Delta_\lambda = (1+\lambda)S^{\mu\nu}\partial_\mu\partial_\nu + \lambda^2 T^\mu\partial_\mu + R$$

is a pencil of 2-nd order operators, but the operator

$$\hat{\Delta} = (1+\hat{w})S^{\mu\nu}\partial_\mu\partial_\nu + \hat{w}^2 T^\mu\partial_\mu + R$$

is an operator of the order 3.

Canonical scalar product on algebra of densities $\mathcal{F}(M)$

Definition

$\mathbf{s}_1 = s_1(x)|Dx|^{\lambda_1}$, $\mathbf{s}_2 = s_2(x)|Dx|^{\lambda_2}$

$$\langle \mathbf{s}_1, \mathbf{s}_2 \rangle = \begin{cases} \int_M s_1(x) s_2(x) |Dx|\,, & \text{if} \quad \lambda_1 + \lambda_2 = 1\,, \\ 0 \text{ if} & \text{if}\, \lambda_1 + \lambda_2 \neq 1\,. \end{cases} \tag{1}$$

Remark. We do not need a volume form to define a scalar product on densities. We suppose by default that integrals are well defined: e.g. M is compact or densities under consideration have compact support

Adjointness of operators with respect to the scalar product

A linear differential operator $\widehat{\Delta}$ acting on the algebra $\mathscr{F}(M)$ has an adjoint operator $\widehat{\Delta}^*$:

$$\langle \Delta \mathbf{s}_1, \mathbf{s}_2 \rangle = \langle \mathbf{s}_1, \Delta^* \mathbf{s}_2 \rangle \quad \left(\widehat{\Delta}^*\right)\Big|_{\hat{w}=\lambda} = \left(\hat{\Delta}\Big|_{\hat{w}=1-\lambda}\right)^* . \qquad (2)$$

$x^* = x$, $\partial_x^* = -p_x$, $\hat{w}^* = 1 - \hat{w}$.
if $\mathbf{s}_1 = s_1(x)|Dx|^{\lambda_1}$ and $\mathbf{s}_2 = s_2(x)|Dx|^{\lambda_2}$ then

$$\lambda_1 \langle \mathbf{s}_1, \mathbf{s}_2 \rangle = (1 - \lambda_2) \langle \mathbf{s}_1, \mathbf{s}_2 \rangle ,$$

$$\langle \hat{w} \mathbf{s}_1, \mathbf{s}_2 \rangle = \langle \mathbf{s}_1, (1 - \hat{w}) \mathbf{s}_2 \rangle ,$$

$$\text{i.e. } \hat{w}^* = 1 - \hat{w}$$

Algebra of densities on $M \neq$ algebra of all functions on $\hat{M}$

$$\mathscr{F}(M) \ni \mathbf{s}(x) = s_1(x)|Dx|^{\lambda_1} + s_2(x)|Dx|^{\lambda_2} + \cdots + s_k(x)|Dx|^{\lambda_k}, \quad |Dx| \sim t$$

$$\mathscr{F}(M) = \{s(x,t) = \sum_\lambda s_\lambda(x) t^\lambda\} \subset C(\hat{M}) \neq C(\hat{M}).$$

Algbera of densities can be identified with subalgbera of functions on x, t **which are polynomial on** t, and it is not the algebra of all (smooth) functions on extended manifold $\hat{M}$,

With an abuse of language, we say that a function $f(x,t)$ is a *polynomial* in t if it is a finite sum of 'monomials' of arbitrary real degrees, $f(x,t) = \sum_\lambda f_\lambda(x) t^\lambda$, $\lambda \in \mathbf{R}$.

Different algebras of functions on $\hat{M} \mapsto$ different structures:

$$\begin{array}{ll} \hat{M} & \text{coordinates } (x^\mu, t) \\ \downarrow & \\ M & \text{coordinates } (x^\mu) \end{array}$$

functions polynomial on t	—all smooth functions on x, t
$\mathscr{F}(M) = \{s(x,t) = s_\lambda(x)t^\lambda\}$	$C(M) = \{s(x,t)\}$
Canonical scalar product. Geometrical constructions invariant with respect to **all** diffeomorphisms	Canonical lifting of projective class on M on affine connection on $\hat{M}$. Sort of "projective" geometry.

Divergence of vector fields

First order operator $=$ vector field $+$ scalar field.
Let $\hat{\mathbf{K}}$ be an arbitrary vector field on extended manifold $\hat{M}$:

for all $\mathbf{s}_1, \mathbf{s}_2 \in \mathscr{F}(M)$, $\hat{\mathbf{K}}(\mathbf{s}_1\mathbf{s}_2) = \hat{\mathbf{K}}(\mathbf{s}_1)\mathbf{s}_2 + \mathbf{s}_1\hat{\mathbf{K}}(\mathbf{s}_2)$, (Leibnitz rule).

$$\hat{\mathbf{K}} = K^\mu \partial_\mu + K^0 \hat{w} = K^\mu \partial_\mu + K^0 t \frac{\partial}{\partial t}\,.$$

we suppose that $\hat{\mathbf{K}}$ does not change a weight of densities.

$$\hat{\mathbf{K}}^* = -K^\mu \partial_\mu - \partial_\mu K^\mu + K^0(1 - \hat{w})$$

Definition: $\operatorname{div} \hat{\mathbf{K}} = -(\hat{\mathbf{K}} + \hat{\mathbf{K}}^*) = \partial_\mu K^\mu - K^0$

It is canonical construction. It does not depende on a choice of coordinates.

Lie derivative of densities—-divergence-less vector field

Let $\hat{\mathbf{K}}$ be vector field on extendend manifold $\hat{M}$,
If $\mathrm{div}\,\mathbf{K} = 0$, i.e. $\hat{\mathbf{K}} = -\hat{\mathbf{K}}^*$,i.e. $\partial_\mu K^\mu - K^0 = 0$, then

$$\hat{\mathbf{K}} = K^\mu \partial_\mu + K^0 \hat{w} = K^\mu \partial_\mu + \partial_\mu K^\mu \hat{w}\,, \quad \hat{K}\big|_{\hat{w}=\lambda} = K^\mu \partial_\mu + \lambda \partial_\mu K^\mu\,.$$

$$\underbrace{\mathbf{K} = K^\mu \partial_\mu}_{\text{vector field on } M} \quad \text{can be canoncally lifted to } \underbrace{\hat{\mathbf{K}} = K^\mu \partial_\mu + \hat{w} \partial_\mu K^\mu}_{\text{vector field on } \hat{M}}$$

Lie derivative $\mathscr{L}_{\mathbf{K}}^{(\lambda)}$ of density of weight λ along vector field $\mathbf{K}$:

$$\mathscr{L}_{\mathbf{K}}^{(\lambda)} (s(x)|Dx|^\lambda) = \left(K^\mu \partial_\mu s(x) + \lambda \partial_\mu K^\mu s(x)\right) |Dx|^\lambda$$

$$\hat{\mathbf{K}} = \hat{\mathscr{L}}_{\mathbf{K}} = K^\mu \partial_\mu + \hat{w} \partial_\mu K^\mu$$

is Lie derivative on algebra $\mathscr{F}(\mathscr{M})$ of all densities.

Second order operator

Let $\hat{\Delta}$ be a second order operator on $\mathscr{F}(M)$.

$$\hat{\Delta} = \underbrace{S^{\mu\nu}(x)\partial_\mu\partial_\nu + \hat{w}B^\mu(x)\partial_\mu + \hat{w}^2C(x)}_{\text{second order derivatives}} + \underbrace{D^\mu(x)\partial_\mu + \hat{w}E(x)}_{\text{first order derivatives}} + F(x).$$

Put normalisation condition $F = \hat{\Delta}(1) = 0$. Adjoint $\hat{\Delta}^* =$

$$\partial_\mu\partial_\nu(S^{\mu\nu}\dots) - \hat{w}^*\partial_\mu(B^\mu + (\dots)) + (\hat{w}^*)^2(C\dots) - \partial_\mu(D^\mu\dots) + \hat{w}^*E.$$

The condition $\hat{\Delta}^* = \hat{\Delta}$ implies that

$$\hat{\Delta} = S^{\mu\nu}\partial_\mu\partial_\nu + \partial_\nu S^{\nu\mu}\partial_\mu + (2\hat{w}-1)\gamma^\mu\partial_\mu + \hat{w}\partial_\mu\gamma^j\mu + \hat{w}(\hat{w}-1)\theta.$$

We have denoted $\gamma^\mu(x) = 2B^\mu(x)$ and $\theta(x) = C(x)$.

Kaluza-Klein mechanism

For self-adjoint operator

$$\hat{\Delta} = \underbrace{S^{\mu\nu}\partial_\mu\partial_\nu + 2\hat{w}\gamma^\mu\partial_\mu + \hat{w}^2\theta}_{\text{second order derivatives on } \hat{M}} + \dots, \qquad \left(\hat{w} = t\frac{\partial}{\partial t}\right),$$

Principal symbol of Δ

$$\hat{S} = \begin{pmatrix} S^{\mu\nu} & \gamma^\mu \\ \gamma^\mu & \theta \end{pmatrix}$$

$$\begin{array}{llll} \hat{\mathbf{K}}\text{ vector field on } \hat{M}, & \hat{M} & \text{coordinates} & (x^\mu, t) \\ \uparrow & \downarrow & & \\ \mathbf{K}\text{ vector field on } M, & M & \text{coordinates} & (x^\mu) \end{array}$$

$$\hat{\mathbf{K}}: \quad \langle \mathbf{K}, \hat{w} \rangle = 0,$$

$$\mathbf{K} = K^\mu\partial_\mu \mapsto \hat{\mathbf{K}} = K^\mu\partial_\mu + \hat{w}\gamma_\mu K^\mu$$

This is Kaluza-Klein mechanism of defining connection

(Transformation rules) γ_μ is connection of this bundle.
($\gamma^\mu = S^{\mu\nu}\gamma_\nu$).

$$\gamma^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu}\left(\gamma^\mu + S^{\mu\nu}\partial_\nu \log J\right),$$

$$\theta' = \theta + 2\gamma^\mu \partial_\mu \log J + \partial_\mu \log J\, S^{\mu\nu}\partial_\nu \log J,$$

where $J = \det\left(\frac{\partial x^{\mu'}}{\partial x^\mu}\right)$ We call θ Brans-Dicke function

Remark $\theta = \gamma^\mu\gamma_\mu + F$ where F is a scalar function. If $F = 0$ then horisontal lifitng (connection) is defined by the null-vector of principal symbol

Additive coordinate in fibre

$$t' = tJ \longrightarrow x^{5'} = x^5 + f(x^\mu)i, \text{ where}$$
$$f(x^\mu) = \log J = \log\left(\det\left(\frac{\partial x^{\mu'}}{\partial x^\mu}\right)\right)$$

$$S^{\mu\nu} \sim \text{Riemannian metric } G^{\mu\nu}$$

γ_μ — — — connection on volume forms $\sim A_\mu$ — — — electromagnetic field

Uniqueness

Theorem

Let Δ be second order operator acting on densities of weight λ_0, where $\lambda_0 \neq 0, 1, 1/2$.
Then there exists a unique self-adjoint operator $\hat{\Delta}$, ($\hat{\Delta}^* = \hat{\Delta}$) such that

$$\hat{\Delta}\Big|_{\hat{w}=\lambda_0} = \Delta \,,$$

and it is obeyed the normalisation condition $\hat{\Delta}(1) = 0$.

In other words there exists unique self-adjoint normalised pencil of second order operators which passes through a given operator (if $\lambda_0 \neq 0, 1, 1/2$).

Application of uniqueness theorem

Does there exist equivariant map $\sigma_{\mu,\lambda}$ between operators on densities of weight λ and operators on densities of weight μ?

Answer on this question is an easy exercise for operators of first order.
It is beautiful and not trivial for operators of second order.
In general case it is negative for operators of order 3 and higher order.

Operators of order $n = 1$

Let $\hat{\mathbf{A}} = A^\mu \partial_\mu + A(x)$ be a first order operator on densities of weight λ. Naive answer $\sigma_{\mu\lambda}(\hat{\mathbf{A}}) = \hat{\mathbf{A}}$ is wrong! (It does not survive under changing of coordinates).

$$\text{Note that } \hat{\mathbf{A}} = \underbrace{\left(A^\mu \partial_\mu + \lambda \partial_\mu A^\mu\right)}_{\text{Lie derivative } \mathscr{L}^{(\lambda)}_{\mathbf{A}}} + \underbrace{\left(A(x) - \lambda \partial_\mu A^\mu(x)\right)}_{\text{scalar function } S(x)} .$$

$$\sigma_{\mu\lambda}(A^\mu \partial_\mu + A) = \sigma_{\mu\lambda}\left(\mathscr{L}^{(\lambda)} + S(x)\right) = \mathscr{L}^{(\mu)}_{\mathbf{A}} + S(x) =$$

$$A^\mu \partial_\mu + (\mu - \lambda)\partial_\mu A^\mu + A .$$

Operators of order n=2

Let $\Delta = A^{\mu\nu}\partial_\mu\partial_\nu + A^\mu\partial_\mu + A$. Consider the normalised self-adjoint operator $\hat{\Delta} = \hat{\Delta}_{S,\gamma^\mu,\theta}$, which passes through this operator: $\Delta = A^{\mu\nu}\partial_\mu\partial_\nu + A^\mu\partial_\mu + A = \left(\hat{\Delta}_{S,\gamma^\mu,\theta}\right)_{\hat{w}=\lambda} =$

$$\left(S^{\mu\nu}\partial_\mu\partial_\nu + \partial_\nu S^{\nu\mu}\partial_\mu + (2\hat{w}-1)\gamma^\mu\partial_\mu + \hat{w}\partial_\mu\gamma^j\mu + \hat{w}(\hat{w}-1)\theta\right)_{\hat{w}=\lambda},$$

$$\begin{cases} S^{\mu\nu} = A^{\mu\nu} \\ \partial_\nu S^{\nu\mu} + (2\lambda-1)\gamma^\mu = A^\mu \\ \lambda\partial_\mu\gamma^\mu + \lambda(\lambda-1)\theta = A \end{cases}$$

These relations uniquely define the operator pencil $\hat{\Delta}_{S,\gamma.\theta}$. Hence

$$\sigma_{\mu,\lambda}(\Delta) = \left(\hat{\Delta}_{S.\gamma,\theta}\right)\Big|_{\hat{w}=\mu}$$

(We draw the pencil through the operator then fix an another weight).)

Answer

$$\sigma_{\lambda,\mu}(A^{\mu\nu}\partial_\mu\partial_\nu + A^\mu\partial_\mu + A) = B^{\mu\nu}\partial_\mu\partial_\nu + B^\mu\partial_\mu + B\,,$$

where

$$\begin{cases} B^{\mu\nu} & = A^{\mu\nu}\,, \\ B^\mu & = \frac{2\mu-1}{2\lambda-1}A^\mu + \frac{2(\lambda-\mu)}{2\lambda-1}\partial_\nu A^{\nu\mu}\,, \\ B & = \frac{\mu(\mu-1)}{\lambda(\lambda-1)}A + \frac{\mu(\lambda-\mu)}{(2\lambda-1)(\lambda-1)}\left(\partial_\nu A^\nu - \partial_\mu\partial_\nu A^{\mu\nu}\right)\,. \end{cases}$$

Special example

Consider the special case when $\Delta = \mathscr{L}^{\lambda}_{\mathbf{X}} \circ \mathscr{L}^{(\lambda)}_{\mathbf{Y}}$. In this case calculate the answer:

$$\mathscr{L}^{\lambda}_{\mathbf{X}} \circ \mathscr{L}^{(\lambda)}_{\mathbf{Y}} = \frac{1}{2}\left(\mathscr{L}^{\lambda}_{\mathbf{X}} \circ \mathscr{L}^{(\lambda)}_{\mathbf{Y}} + \mathscr{L}^{\lambda}_{\mathbf{Y}} \circ \mathscr{L}^{(\lambda)}_{\mathbf{X}}\right) + \frac{1}{2}\left(\mathscr{L}^{\lambda}_{\mathbf{X}} \circ \mathscr{L}^{(\lambda)}_{\mathbf{Y}} - \mathscr{L}^{\lambda}_{\mathbf{Y}} \circ \mathscr{L}^{(\lambda)}_{\mathbf{X}}\right)$$

$$= \frac{1}{2}\left(\mathscr{L}^{\lambda}_{\mathbf{X}} \circ \mathscr{L}^{(\lambda)}_{\mathbf{Y}} + \mathscr{L}^{\lambda}_{\mathbf{Y}} \circ \mathscr{L}^{(\lambda)}_{\mathbf{X}}\right) + \mathscr{L}^{(\lambda)}_{[\mathbf{X},\mathbf{Y}]}\,.$$

Draw self-adjoint operator which passes through Δ:

$$\hat{\Delta} = \underbrace{\frac{1}{2}\left(\hat{\mathscr{L}}_{\mathbf{X}} \circ \hat{\mathscr{L}}_{\mathbf{Y}} + \hat{\mathscr{L}}_{\mathbf{Y}} \circ \mathscr{L}_{\mathbf{X}}\right)}_{\text{self-adjoint}} + ? \qquad \underbrace{\hat{\mathscr{L}}_{[\mathbf{X},\mathbf{Y}]}}_{\text{anti-self-adjoint}} \qquad ,$$

$$\left(\hat{\mathscr{L}}_{\mathbf{X}}\right)^{*} = -\hat{\mathscr{L}}_{\mathbf{X}}, \quad ?: \quad ,(?)^{*} = -?, \quad ?\big|_{\hat{w}=\lambda} = 1\,.$$

Special example

Consider the special case when $\Delta = \mathcal{L}^{\lambda}_{\mathbf{X}} \circ \mathcal{L}^{(\lambda)}_{\mathbf{Y}}$. In this case calculate the answer:

$$\mathcal{L}^{\lambda}_{\mathbf{X}} \circ \mathcal{L}^{(\lambda)}_{\mathbf{Y}} = \frac{1}{2}\left(\mathcal{L}^{\lambda}_{\mathbf{X}} \circ \mathcal{L}^{(\lambda)}_{\mathbf{Y}} + \mathcal{L}^{\lambda}_{\mathbf{Y}} \circ \mathcal{L}^{(\lambda)}_{\mathbf{X}}\right) + \frac{1}{2}\left(\mathcal{L}^{\lambda}_{\mathbf{X}} \circ \mathcal{L}^{(\lambda)}_{\mathbf{Y}} - \mathcal{L}^{\lambda}_{\mathbf{Y}} \circ \mathcal{L}^{(\lambda)}_{\mathbf{X}}\right)$$

$$= \frac{1}{2}\left(\mathcal{L}^{\lambda}_{\mathbf{X}} \circ \mathcal{L}^{(\lambda)}_{\mathbf{Y}} + \mathcal{L}^{\lambda}_{\mathbf{Y}} \circ \mathcal{L}^{(\lambda)}_{\mathbf{X}}\right) + \mathcal{L}^{(\lambda)}_{[\mathbf{X},\mathbf{Y}]}\,.$$

Draw self-adjoint operator which passes through Δ:

$$\hat{\Delta} = \underbrace{\frac{1}{2}\left(\hat{\mathcal{L}}_{\mathbf{X}} \circ \hat{\mathcal{L}}_{\mathbf{Y}} + \hat{\mathcal{L}}_{\mathbf{Y}} \circ \mathcal{L}_{\mathbf{X}}\right)}_{\text{self-adjoint}} + \frac{2\hat{w}-1}{2\lambda-1} \underbrace{\hat{\mathcal{L}}_{[\mathbf{X},\mathbf{Y}]}}_{\text{anti-self-adjoint}}\,,$$

$$\left(\hat{\mathcal{L}}_{\mathbf{X}}\right)^{*} = -\hat{\mathcal{L}}_{\mathbf{X}}, \quad \left(\frac{2\hat{w}-1}{2\lambda-1}\right)^{*} = -\frac{2\hat{w}-1}{2\lambda-1}, \quad \frac{2\hat{w}-1}{2\lambda-1}\Big|_{\hat{w}=\lambda} = 1\,.$$