Thick morphisms and spinors

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The talk is based on the work with Ted Voronov

Contents

Papers that talk is based on are

[1]. H.M.Khudaverdian, Th.Voronov “Thick morphisms, higher Koszul brackets, and L_∞-algebroids”, math-arXiv:1808.10049

[2]. H.M.Khudaverdian, Th.Voronov “Thick morphisms of supermanifolds, quantum mechanics and spinor representation”, math-arXiv:1909.00290

[3] Th. Voronov, *Nonlinear pullback on functions and a formal category extending the category of supermanifolds]*, arXiv: 1409.6475

[4] Th. Voronov, *Microformal geometry*, arXiv: 1411.6720

Abstract...

For an arbitrary morphism $\varphi\colon\ M \to N$ of (super)manifolds, the pull-back $\phi^* C^\infty(N) \to C^\infty(M)$ is a linear map of space of functions. Moreover it is homomorphism of algebra $C^\infty(N)$ into algebra $C^\infty(M)$.
In 2014 Voronov have introduced *thick morphisms* of (super)manifolds which define generally non-linear pull-back of functions. This construction was introduced as an adequate tool to describe L_∞ morphisms of algebras of functions provided with the structure of homotopy Poisson algebra.

Thick morphism $\Phi = \Phi_S\colon \quad M \Rrightarrow N$ can be defined by the "action" $S(x,q)$, where x are local coordinates on M and q are coordinates of momenta in T^*N. The pull-back of thick morphism $\Phi_S^*\colon\ C^\infty(N) \to C^\infty(M)$, is non-linear map in the case if the action $S(x,q)$ is not linear over q.
In this approach we come to fundamental concepts of Quantum Mechanics.
In particular thick morphisms with quadratic action give naturally the spinor representation.

Let M_1, M_2 be two (super)manifolds.

$$\underbrace{M_1}_{x^i\text{- loc.coord.}}\,, \qquad \underbrace{M_2}_{y^a\text{- loc.coord.}}\,,$$

Consider also cotangent bundles T^*M_1 and T^*M_2.

$$\underbrace{T^*M_1}_{x^i, p_j\text{- loc.coord.}}\,, \qquad \underbrace{T^*M_2}_{y^a, q_b\text{- loc.coord.}}\,,$$

p_i are components of momenta which are conjugate to x^i, respectively q_a are components of momenta which are conjugate to y^a.

Remark We consider even and odd coordinates, i.e. M_1, M_2 are supermanifolds; parity of any coordinate coincide with the parity of corresponding component of momenta:

$$p(p_i) = p(x^i)\,, \quad p(q_b) = p(y^b)\,.$$

Definition of thick morphism. (T.Voronov)

M_1, M_2–two (super)manifolds
Consider symplectic manifold $T^*M_1 \times (-T^*M_2)$
equipped with canonical symplectic structure

$$\omega = \omega_1 - \omega_2 = \underbrace{dp_i \wedge dx^i}_{\text{coord. on } T^*M_1} - \underbrace{dq_a \wedge dy^a}_{\text{coord. on } T^*M_2}$$

Function $S = S(x, q)$—action
It defines Lagrangian surface $\Lambda_S \subset T^*M_1 \times (-T^*M_2)$:

$$\Lambda_S = \left\{ (x, p, y, q) : \quad p_i = \frac{\partial S(x,q)}{\partial x^i}, y^b = \frac{\partial S(x,q)}{\partial q_b} \right\}$$

Lagrangian surface—canonical relation—thick morphism

Lagr. surf. Λ_S is canon. relation Φ_s in $T^*M_1 \times (-T^*M_2)$

$$(x^i, p_j) \sim_S (y^a, q_b) \leftrightarrow (x^i, p_j, y^a, q_b) \in \Lambda_S\,, (\Phi_S = \sim_S)\,.$$

$$\Phi = \Phi_s \text{ is a thick morphism } M_1 \Rrightarrow M_2$$

It defines pull-back Φ_S^* of functions

$$\Phi_S^* \colon \mathfrak{M}_2 = C(M_2) \to \mathfrak{M}_1 = C(M_1)\,,$$

such that for every function $g = g(y) \in \mathfrak{M}_2$,

$$f = f(x) = (\Phi_S^* g)(x) \colon \Lambda_f = \Phi_S \circ \Lambda_g\,,$$

where Λ_f, Λ_g are Lagrangian surfaces, graphs of df, dg in T^*M_1, T^*M_2.

Explicit expression

Thick morphism $M \Rrightarrow N$ defines the pull-back
$\Phi_g^*: \ C^\infty(N) \to C^\infty(M)$, such that

$$\Phi_S^* g = f(x) = g(y) + S(x,q) - y^a q_a \,,$$

where y^a and q_a are defined from the equations

$$y^a = \frac{\partial S(x,q)}{\partial q_a} \,, \qquad q_a = \frac{\partial g(y)}{\partial y^a}$$

We see that $\Lambda_f = \Phi_S \circ \Lambda_g$ since

$$p_i = \frac{\partial f}{\partial x^i} = \frac{\partial}{\partial x^i} \left(g(y) + S(x,q) - y^a q_a \right) = \frac{\partial S(x,q)}{\partial x^i} \,.$$

Thick morphism is usual map if $S(x,q) = S^a(x)q_q$

Example

Generating function $S = S^a(x)q_a$

$$(\Phi_S^* g)(x) = g(y) + S(x,q) - y^a q_a = g(y) + \underbrace{(S^a(x) - y^a)}_{\text{vanishes}} q_a = g(S^a(x))$$

Thick morphism $M_1 \overset{\Phi_s}{\Rrightarrow} M_2$ is usual morphism $M_1 \overset{y^a = S^a(x)}{\rightarrow} M_2$.

Thick morphism in general case

In general case the pull-back is non-linear:

$$f(x) = (\Phi_S^* g)(x) = \left(S(x,q) + g(y) - y^i q_i\right)\Big|_{y=\frac{\partial S(x,q)}{\partial q},\, q=\frac{\partial g}{\partial y}}\,,$$

Example

$S(x,q) = xq + \frac{1}{2}aq^2$, $g(y) = \frac{1}{2}ky^2$ then $q = ky$, $y = y(x)$ is defined by relation

$$y = \frac{\partial S(x,q)}{\partial q} = x + aq = x + aky \Rightarrow y = \frac{x}{1-ak}\,,$$

and

$$f(x) = \Phi_S^*(g)(x) = S(x,q) + g(y) - yq = \frac{kx^2}{2(1-ak)}\,.$$

Application of thick morphisms: L_∞ morphisms

Consider two homotopy Poisson algebras defined on space of functions $C^\infty(M_i)$ by Hamiltonian Q_i $(i = 1, 2)$
We say that Hamiltonians Q_1, Q_2 are connected by the action $S(x, q)$ if

$$Q_1\left(x^i, p_j = \frac{\partial S(x,q)}{\partial x^j}\right) \equiv Q_2\left(y^a = \frac{\partial S(x,q)}{\partial q_a}, q_b\right)$$

(x^i-coordinates on M_1 and q_a momenta on fibers of T^*M_2)

Theorem

The pull-back Φ_S^* of the thick morphism Φ_S is L_∞ morphism of homotopy Poisson algebra $(C^\infty(M_2), Q_2)$ on homotopy Poisson algebra $(C^\infty(M_1), Q_1)$. (Th.Voronov, 2014)

What is it ‘Homotopy Poisson algebras’. Recalling

Let M be a supermanifold, and let $Q = Q(x,p)$ be an odd Hamiltonian, odd function defined on cotangent bundle T^*M This Hamiltonian defines hmotopy Poisson bracket on algebra of functions $C^\infty(M)$. The chain of brackets ican be defined by

$$\{f_1\} = (Q, f_1)\big|_M\,, \quad \{f_1, f_2\} = ((Q, f_1), f_2)\big|_M\,,$$

$$\{f_1, f_2, f_3\} = (((Q, f_1), f_2)\, f_3)\big|_M\,, \quad \text{and so on:}$$

$$\{f_1, \ldots, f_n\} = \underbrace{(\ldots(}_{n-\text{times}}\; Q, f_1), \ldots, f_n)\big|_M\,,$$

$(\ ,\ )$— canonical even Poisson bracket on T^*M
Q obeys condition $(Q,Q) = 0$—Jacobi identity.
The chain of brackets $\{f_1, \ldots, f_n\}$ becomes an usual odd Poisson) bracket if Hamiltonian Q is quadratic on momenta.

The function $S = S(x, q)$ which defines thick morphism $\Phi_S \colon M \Rrightarrow N$ we call in this paper'action'[1]. Why?

Let $H = H(x, p)$ be Hamiltonian defined in cotangent bundle T^*M and let $\mathfrak{S} = \mathfrak{S}(t, x, y)$ be the action of classical mechanics for the path $x(\tau)$, $0 \leq \tau \leq t$ which obeys equations of motion, starts at the point x at $\tau = 0$, and ends at the point y at time $\tau = t$.

[1] In the pioneer works of T.Voronov, where thick morphism was suggested, this function was called just "generating function"

Its Legendre transform

$$S(t,x,q) = yq - \mathfrak{S}(t,x,y)\,,\text{where } y = \frac{\partial \mathfrak{S}}{\partial q}\,.$$

It obeys Hamilton Jacobi equation

$$S(x,q,t)\colon \begin{cases} \frac{\partial S}{\partial t} = H\left(\frac{\partial S}{\partial q}, q\right) \\ S(x,q)\big|_{t=0} = xq \end{cases}$$

Denote $S(t,x,q) = \exp tH$.

Example

free particle

$$H_{\mathrm{free}} = \frac{p^2}{2m}$$

$$\exp tH_{\mathrm{free}}:$$

$$\mathfrak{S}(t,x,y) = \frac{m(y-x)^2}{2t}\,, \quad S(t,x,q) = xq + \frac{q^2 t}{2m}\,,$$

Example

harmonic oscillator

$$H = \frac{p^2 + x^2}{2},$$

$$\exp tH_{\text{oscillator}}:$$

$$\mathfrak{S}(t, x, y) = \frac{x^2 + y^2}{2} \operatorname{cotan} t - \frac{yx}{\sin t}, \quad S(t, x, q) = xq \cos t + \frac{x^2 + q^2}{2} \tan t,$$

Theorem

Let action $S(t,x,q)$ is an exponent of Hamiltonian H:

$$S(t,x,q) = \exp tH$$

Consider the one-parametric group of thick morphism $\Phi_t\colon M \Rrightarrow M$ generated by $S(t,x,q)$. For an arbitrary function $g = g(x)$ consider

$$f_t(x) = \Phi_t^*(g)$$

The function $f_t(x)$ obeys the Hamilton-Jacobi equation:

$$\frac{\partial f_t(x)}{\partial t} = H\left(x, \frac{\partial f}{\partial x}\right), \quad f_t(x)\big|_{t=0} = g(x).$$

Quantum thick morphisms

$S_\hbar(x,q)$-quantum action, power series in q and $\hbar$

The corresponding `quantum thick morphism` performs the pull-back:

$$\Phi^{*\text{quant.}}_{S_\hbar}(w)(x) = \int_{T^*N} e^{\frac{i}{\hbar}\left(S_\hbar(x,q) - y^i q_i\right)} w(y) DqDy \,.$$

$DqDp$ is invariant Lioville measure on T^*M

Quantum thick morphisms → classical thick morphisms

One can see this using stationary phase method:
For $w_\hbar = e^{\frac{i}{\hbar} g(y)}$

$$\lim_{\hbar \to 0} \left[\frac{\hbar}{i} \left(\log \left(\Phi^{*\text{quant.}}_{S_\hbar} (w_\hbar) \right) \right) \right] = \lim_{\hbar \to 0} \left[\frac{\hbar}{i} \left(\log \left(e^{\frac{i}{\hbar} \left(g(y) + S_\hbar(x,q) - y^i q_i \right)} \right) \right) \right] =$$

$$g(y_0) + S(x, q^0) - y_0^i q_i^0 \,,$$

where $y_0 = y_0(x)$ and $q^0 = q^0(x)$ are defined (depending on x) by the sationary point condition: $y_0^i = \frac{\partial S(x,q)}{\partial q_i} | q_i = q_i^0$ and $q_i^0 = \frac{\partial g(y)}{\partial y^i} |_{y^i = y_0^i}$, and $S(x,q) = \lim_{\hbar \to 0} S_\hbar(x,q)$.
We come to the classical thick morphism:

$$\lim_{\hbar \to 0} \left[\frac{\hbar}{i} \left(\log \left(\Phi^{*\text{quant.}}_{S_\hbar} \left(e^{\frac{i}{\hbar g(y)}} \right) \right) \right) \right] = \Phi^{\text{class}}_{S_0} (g)(x) \,.$$

Legendre transform → Fourier transform

Legendre transform is quasiclassics of Fourier transform:
Legendre: $g(p) = G(x) = px$ such that $G'(x) = p$

$$e^{\frac{i}{\hbar}(G(x)-px)}dx \approx e^{\frac{i}{\hbar}g(p)}.$$

Classical thick morphisms — Hamilton Jacobi equation
Quantum thick morphisms— Shrodinger equation:

What is a spinor

Thick morphism acts on functions on n variables. On the other hand it is defined by an action $S(x, q)$ which depends on $2n$ variables.
This strongly resembles spinor representation if one recalls that the spinor representation (in the orthogonal or symplectic settings) can be seen as action of transformations of a large space on objects such as functions or half-forms that live on a (half-dimensional) maximally isotropic subspace.

Symplectic (orthogonal) spinor is a function on space of half-dimensions, which transforms under the action of spinor group

V-vector space, $\dim V = N$, $X = V \oplus V^*$, $\dim X = 2N$.

$$X \ni \mathbf{A} = \begin{pmatrix} a^i \\ \alpha_j \end{pmatrix} \longrightarrow h_{\mathbf{A}} = a^i \hat{p}_i + \alpha_i \hat{q}^i = \frac{\hbar}{i} a^i \frac{\partial}{\partial x^i} + \alpha_j x^j \,,$$

$$X \ni \mathbf{B} = \begin{pmatrix} b^i \\ \beta_j \end{pmatrix} \longrightarrow h_{\mathbf{B}} = b^i \hat{p}_i + \beta_j \hat{q}^j = \frac{\hbar}{i} b^i \frac{\partial}{\partial x^i} + \beta_j x^j \,,$$

$\mathbf{A}, \mathbf{B}$-vectors in $2N$-dimensional space $\to$ $h_{\mathbf{A}}, h_{\mathbf{B}}$ operators on space of functions on N variables.
Symplectic scalar product $\to$ commutators

$$\langle \mathbf{A}, \mathbf{B} \rangle = a^i \beta_i - \alpha_j b^j = \frac{i}{\hbar} [h_{\mathbf{A}}, h_{\mathbf{B}}]$$

$$\left([\hat{p}_i, \hat{p}_j] = [\hat{q}^i, \hat{q}^j] = 0 \,, [\hat{p}_i, \hat{q}^j] = \frac{\hbar}{i} \right)$$

$$h_{\mathbf{A}} \to \mathfrak{O}^{-1} h_{\mathbf{A}} \mathfrak{O} = h_{\mathbf{A}'}\,,\; h_{\mathbf{B}} \to \mathfrak{O}^{-1} h_{\mathbf{B}} \mathfrak{O} = h_{\mathbf{B}'}\,,$$

$$\mathbf{A} \longrightarrow \mathbf{A}' = g(\mathbf{A})\,,\, \mathbf{B} \longrightarrow \mathbf{B}' = g(\mathbf{B})\,,$$

$$\langle \mathbf{A}, \mathbf{B} \rangle = \frac{i}{\hbar}[h_{\mathbf{A}}, h_{\mathbf{B}}] = \frac{i}{\hbar}\left[\mathfrak{O}^{-1} h_{\mathbf{A}} \mathfrak{O}, \mathfrak{O}^{-1} h_{\mathbf{B}} \mathfrak{O}\right] = \frac{i}{\hbar}[h_{\mathbf{A}'}, h_{\mathbf{B}'}] = \langle \mathbf{A}', \mathbf{B}' \rangle\,.$$

This transformation preserves symplectic scalar product.

spinor group $\ni \mathfrak{O} \to g \colon \mathbf{A} \to \mathbf{A}'$, $g \in$ symplectic group $Sp(n)$

Where are usual spinors?

$$SO(N) = Sp(-N)$$

We come to usual (orthogonal spinors) changing a parity

symplectic group $Sp(n)$ – – – – – Orthogonal group $O(n)$

$X = V \oplus V^*$	– – –	$\Pi X = \Pi V \oplus \Pi V^*$
symplectic space	– – –	Euclidean space
linear operator $h_{\mathbf{A}}$	– – –	linear operator $\gamma_{\mathbf{A}}$
symplectic group $Sp(N)$	– – –	Orthogonal group $O(N)$
acting on space of functions	– – –	acting on space of functions
of commuting coordinates	– – –	of anticommuting coordinates
$x_1, \ldots, x_N$	– – –	$\xi_1, \ldots, \xi_N$
Spinor representation	– – –	Spinor representation
is infinite dimensional	– – –	is finite dimensional

In the symplectic case spinors called metaplectic spinors.

Spinor group and thick morphisms

Return to thick morphisms

Spinor group $\{\mathfrak{O}\}$ can be defined as subgroup of qiantum thick morphisms corresponding to quadratic Hamiltonians.

Let M, N be two (super)manifolds. Recall that the classical action $S = S(x, q)$ connects Hamiltonian H_M on T^*M with Hamiltonian H_N on T^*N if

$$H_M\left(x, \frac{\partial S(x,q)}{\partial x}\right) \equiv H_N\left(\frac{\partial S(x,q)}{\partial q}, q\right).$$

Let $\Delta_M = H_M(\hat{x}, \hat{p})$ be a linear operator on M =the quantum Hamiltonian (operator depending on $\hat{x} = x$ and $\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$, and respectively let $\Delta_N = H_N(\hat{y}, \hat{y})$ be a linear operator on N =the quantum Hamiltonian (operator depending on $\hat{y} = y$ and $\hat{q} = \frac{\hbar}{i}\frac{\partial}{\partial y}$,

Definition

We say that the quantum thick morphism $\Phi_{S_\hbar}$ $M \Rrightarrow N$ connects operators Δ_M and Δ_N if the pull-back Φ_{S_h} of quantum thick morphism commutes with these operators. i.e.

$$\Delta_M \circ \Phi^*_{S_\hbar} = \Phi^*_{S_\hbar} \circ \Delta_N \,, \quad \left(\Delta_N = \left(\Phi^*_{S_\hbar} \right)^{-1} \circ \Delta_M \Phi^*_{S_\hbar} \right)$$

Quantum morpshims $\to$ classical morphisms

Theorem

Let $S_\hbar(x,q)$ be a quantum action such that quantum thick morphism $\Phi_{S_\hbar}$ connects quantum Hamiltonians Δ_M and Δ_N. then

- *classical thick morphism Φ_{S_0} defined by classical action $S_0(x,q) = \lim_{\hbar\to 0} S_\hbar$ connects classical Hamiltonians H_M and H_N (symbols of operators Δ_M and Δ_N).*
- *If Δ_M and Δ_N are operators, such that Hamiltonians (their symbols) H_M, H_N are linear then the condition that quantum thick morphism $\Phi_{S_\hbar}$ connects quantum Hamiltonians $\hat{H}_M$ and $\hat{H}_N$ does not depend on $\hbar$; in particular the condition that classical action connects two linear classical Hamiltonians is equivalent to the condition quantum version.*

Definition of spinor group in terms of thick morphisms

To define spinor group we have to consider thick morphisms corresponding to quadratic Hamiltonians/

Definition

Spinor group is the group of thick diffeomorphisms Φ_S corresponding to quadratic Hamiltonians.