Riemannian Geometry

Coursework 2020

`Here are the solutions of the coursework.`

`The discussions of the coursework will appear on webpage and I plan to send students individual comments.`

Solutions

1

Consider a surface M, the upper sheet of the cone

$$\mathbf{r}(h,\varphi)\colon \begin{cases} x = h\cos\varphi \\ y = h\sin\varphi \\ z = 2h \end{cases}, \qquad 0 \le \varphi < 2\pi\,,\; h > 0\,. \tag{1}$$

Calculate the Riemannian metric G on this surface induced by the Euclidean metric in $\mathbf{E}^3$ in coordinates (h,φ).

Show that this surface is locally Euclidean by giving an example of local coordinates (u,v), which are Euclidean coordinates.

Find the length of the shortest curve which belongs to the surface M, starts at the point $(h_0, 0, 2h_0)$ and ends at the point $(-h_0, 0, 2h_0)$.

[3 marks]

We have $G_M = (dx^2+dy^2+dz^2)\big|_M = (\cos\varphi dh - h\sin\varphi d\varphi)^2 + (\sin\varphi dh + h\cos\varphi d\varphi)^2 + 4dh^2 = 5dh^2 + h^2 d\varphi^2$.

Under changing of coordinates

$$\begin{cases} u = \sqrt{5}h\cos\frac{\varphi}{\sqrt{5}} \\ v = \sqrt{5}h\sin\frac{\varphi}{\sqrt{5}} \end{cases} \tag{1.1}$$

we come to locally Euclidean coordinates: $du^2 + dv^2 = x = 5dh^2 + h^2d\varphi^2$. Hence the surface is locally Euclidean.

Unroll the surface of the upper sheet of the cone for $0 < h < h_0$. (We can do it since conical surface is locally Euclidean.) (`You can see below another more 'pure' solution based on locally Euclidean coordinates (1.1).`)

We will come to the sector of the interior of the circle with radius $R = \sqrt{4{h_0}^2 + h^2} = h_0\sqrt{5}$, and with the angle

$$\theta = \frac{\text{length of the circle}}{\text{radius of the unfolded sector}} = \frac{2\pi h}{R} = \frac{2\pi h_0}{h_0\sqrt{5}} = \frac{2\pi}{\sqrt{5}}\,.$$

The antipodal points at the conical surface, i.e. the points (h,φ), $(h,\varphi+\pi)$ will be presented by the points which are on the cicle of the unfolded csurface, such that the angle between them will be $\frac{\theta}{2}$. The distance between these points is

$$d = 2R\sin\frac{\theta}{4} = 2h_0\sqrt{5}\sin\frac{2\pi}{4\sqrt{5}} = 2h_0\sqrt{5}\sin\frac{\pi}{2\sqrt{5}}\,. \tag{1.2}$$

Sure one can come to this answer in a more 'pure' way:

We proved that the surface of cone is locally Euclidean and showed that in coordinates (1.1) Riemannian metric is Euclidean

$$G = du^2 + dv^2 = 5dh^2 + h^2 d\varphi^2\,, \qquad \begin{cases} u = \sqrt{5}h\cos\frac{\varphi}{\sqrt{5}} \\ u = \sqrt{5}h\sin\frac{\varphi}{\sqrt{5}} \end{cases}.$$

Point $A = (h_0, 0, 2h_0)$ has coordinates

$$(u_0, v_0) = \left(h_0\sqrt{5}, 0\right)$$

since $\cos 0 = 1$ and $\sin 0 = 0$ and point $B = (-h_0, 0, 2h_0)$ has coordinates

$$(u_0, v_0) = \left(h_0\sqrt{5}\cos\frac{\pi}{\sqrt{5}}, h_0\sqrt{5}\sin\frac{\pi}{\sqrt{5}}\right)$$

The length of the shortest curve between points A and B is equal to

$$\sqrt{(u_1-u_0)^2+(v_1-v_0)^2} = \sqrt{\left(\sqrt{5}h_0\cos\frac{\pi}{\sqrt{5}} - \sqrt{5}h_0\right)^2 + \left(\sqrt{5}h_0\sin\frac{\pi}{\sqrt{5}} - 0\right)^2} =$$

$$\sqrt{5}h_0\sqrt{\left(\cos\frac{\pi}{\sqrt{5}} - 1\right)^2 + \sin^2\frac{\pi}{\sqrt{5}}} = \sqrt{5}h_0\cdot 2\sin\frac{\pi}{\sqrt{5}}$$

We came to the same answer (1.2) ■

2

Consider a sphere S^2 of the radius a in spherical coordinates

$$\mathbf{r}(\theta,\varphi)\colon \begin{cases} x = a\sin\theta\cos\varphi \\ y = a\sin\theta\sin\varphi \\ z = a\cos\theta \end{cases}, \qquad 0 \le \varphi < 2\pi\,,\ 0 < \theta < \pi\,. \tag{1}$$

Calculate the Riemannian metric G on this surface induced by the Euclidean metric in $\mathbf{E}^3$ in spherical coordinates (θ,φ).

Give an example of non-identical transformation which preserves the metric.

Consider two points $A = (a\sin\theta_0, 0, a\cos\theta_0)$ $B = (-a\sin\theta_0, 0, a\cos\theta_0)$ *on this sphere. Calculate the length of the arc of the latitude* $\begin{cases}\varphi = t\\ \theta = \theta_0\end{cases}$ *which connects this points.*

Explain why this is not the shortest curve on the sphere which connects points A *and* B*.*

Give an argument explaining why sphere is not locally Euclidean.

[3 marks]

We have $G_{S^2} = (dx^2 + dy^2 + dz^2)\big|_{S^2} =$

$$(-a\sin\theta\sin\varphi d\varphi + a\cos\theta\cos\varphi d\theta)^2 + (a\sin\theta\cos\varphi d\varphi + a\cos\theta\sin\varphi d\theta)^2 + (-a\sin\theta d\theta)^2 =$$

$$a^2\sin^2\theta\sin^2\varphi d\varphi^2 + a^2\cos^2\theta\cos^2\varphi d\varphi^2 + a^2\sin^2\theta\cos^2\varphi d\varphi^2 + a^2\cos^2\theta\sin^2\varphi d\varphi^2 + + a^2\sin^2\theta d\theta^2 =$$

$$a^2\sin^2\theta d\varphi^2 + a^2\cos^2\theta d\theta^2 + a^2\sin^2\theta d\theta^2 = a^2 d\theta^2 + a^2\sin^2\theta d\varphi^2\,. \tag{1}$$

Consider transformation, rotation on angle φ_0 around axis OZ: $F\colon \begin{cases}\theta' = \theta\\ \varphi' = \varphi + \varphi_0\end{cases}$ This transformation is obviously isometry, since it does not change the Riemannian metric (1):

$$a^2 d\theta'^2 + a^2\sin^2\theta' d\varphi'^2 = a^2 d\theta^2 + a^2\sin^2\theta d\varphi^2\,.$$

This arc is half of the circle of the radius $r = a\sin\theta_0$. Hence its length of is equal to $L = \pi a\sin\theta_0$.

If we consider another arc- the arc of the circle which passes throug the poins A, B and the North pole, * we see that the length of this arc is equal to $L' = 2a\theta_0$. We see that $L > L'$:

$$L = \pi a\sin\theta_0 \geq 2a\theta_0\,.$$

They are equal if $\theta = \pi/2$, i.e. the curve L becomes equator.

Remark This answer has the following interpretation: great circle is geodesic, and the arc of latitude becomes geodesic if $\theta = \frac{\pi}{2}$. ■

3

* this is the arc of great circle

Recall that the Riemannian metric on the sphere of radius R in the stereographic coordinates is expressed by the formula

$$G_{\text{stereogr.}} = \frac{4R^4(du^2+dv^2)}{(R^2+u^2+v^2)^2}\,.$$

a) Give an example of a non-identity transformation of coordinates u, v such that it preserves this metric.

b) Give an example of a non-linear transformation of coordinates u, v such that it preserves this metric.

(Hint:You may find this transformation considering transformations of the sphere.)

c) Find the length of the line $v = au$ in $\mathbf{R}^2$ with respect to this metric.

Why the length of this curve does not depend on the parameter a?

[4 marks]

a) An arbitrary orthogonal transformation of $\mathbf{E}^3$ (e.g. an arbitrary rotation, or transformation $\mathbf{r} \mapsto -\mathbf{r}$) transforms sphere $x^2 + y^2 + z^2 = R^2$ onto itself without changing the metric of sphere . They induce corresponding transformations of the stereographic coordinates u, v:

$$\begin{cases} u = \frac{Rx}{R-z} \\ v = \frac{Ry}{R-z} \end{cases}, \quad \begin{cases} x = \frac{2uR^2}{u^2+v^2+R^2} \\ y = \frac{2vR^2}{u^2+v^2+R^2} \\ z = R\frac{u^2+v^2-R^2}{u^2+v^2+R^2} \end{cases}.$$

E.g. rotations along axis z induces linear transformation: we come to linear transformation of coordinates u, v: $\begin{cases} u \mapsto u\cos\theta + v\sin\theta \\ v \mapsto -u\sin\theta + v\cos\theta \end{cases}$.

b) To find non-lnear transformation take an arbitrary orthogonal transformation of $\mathbf{E}^3$, which does not preserve the z-axis. This transformation evidently is isometry of the sphere, and it will induce non-linear transformation of stereographic coordinates. As an example of such transformataion one may consider transformation $\mathbf{r} \mapsto -\mathbf{r}$, $(x, y, z) \mapsto (-x, -y, -z)$. This transformation sends an arbitrary point of sphere to antipodal and it obviously preserves the metric. Using formulae for stereographic coordinates we come to:

$$\begin{cases} u = \frac{Rx}{R-z} \mapsto \frac{-Rx}{R+z} = -\frac{uR^2}{u^2+v^2} \\ v = \frac{Ry}{R-z} \mapsto \frac{-Ry}{R+z} = -\frac{vR^2}{u^2+v^2} \end{cases}$$

This is an inversion (up to a sign). In complex coordinates it looks as $z = u+iv$, $z \mapsto -\frac{R^2}{\bar{z}}$). (One may consider also another transformation: rotation along axis Oy on the angle $\pi/2$: $(x, y, z) \mapsto (y, z, -y)$, respectively $u = \frac{Rx}{R-z} \mapsto \frac{Rx}{R+y} = \ldots$ and $v = \frac{Ry}{R-z} \mapsto \frac{Rz}{R+y} = \ldots.$)

`Another solution` In complex coordinates $z = u + iv$ the metric is

$$G = \frac{4R^4(du^2 + dv^2}{(R^2 + u^2 + v^2)^2)} = \frac{4R^4 dz d\bar{z}}{(R^2 + z\bar{z})^2} .$$

One can see that $z = \frac{R^2}{w}$, i.e. is isometry.

c) An arbitrary line passing through the origin is the isometric image of a great circle pasing through North and South pole of the sphere. Length of any great circle is equal to $2\pi R$, hence the length of the line is equal to $2\pi R$, this is the length of the great circle of the sphere. It obviously does not depend on a.

Remark One can calculate the length of this line straightforwardly using brute force calculations: The length of the line $v = au$ with respect to the metric (1) is equal to $L =$

$$\int \sqrt{\left(\frac{4R^4}{(R^2 + u^2 + v^2)^2}\right)(u_t^2 + v_t^2)dt} = \int_{-\infty}^{\infty} \sqrt{\left(\frac{4R^4}{(R^2 + u^2 + a^2u^2)^2}\right)(a^2 + 1)du} =$$

$$2R \int_{-\infty}^{\infty} \frac{dz}{1 + z^2} = 2\pi R , \quad \text{where } z = R\sqrt{a^2 + 1}u .$$

■

4.

Evaluate the area of the part of the sphere of radius $R = 1$ between the planes given by equations $2x + 2y + z = 1$ and $2x + 2y + z = 2$.

[2 marks]

These planes are parallel. The normal equations of these planes are $\frac{2}{3}x + \frac{2}{3}x + \frac{2}{3}y + \frac{2}{3}z = \frac{1}{3}$ and $\frac{2}{3}x + \frac{2}{3}y + \frac{2}{3}z = \frac{2}{3}$ The first plane is on the distance $\frac{1}{3}$ of the origin, the second is on the distance $\frac{2}{3}$. They both intersect the sphere. The distance between the planes equals to $h = \frac{1}{3}$. The area of the part of the sphere between the planes is $S = 2\pi RH = \frac{2\pi}{3}$. ■

5.

Consider the plane $\mathbf{R}^2$ with standard coordinates (x, y) equipped with Riemannian metric

$$G = (1 + x^2 + y^2)e^{-a^2x^2 - a^2y^2} \left(dx^2 + dy^2\right) .$$

Calculate the total area of this plane.

[1 marks]

Transform the Riemannian metric $G = (1 + x^2 = y^2)e^{-a^2(x^2+y^2)}(dx^2 + dy^2)$ to polar coordinates r, φ: $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$:

$$G = (1 + r^2)e^{-a^2r^2}(dr^2 + r^2 d\varphi^2), \qquad ||g_{\alpha\beta}|| = \begin{pmatrix} (1 + r^2)e^{-a^2r^2} & 0 \\ 0 & r^2(1 + r^2)e^{-a^2r^2} \end{pmatrix},$$

and volume form is equal to $dv = \sqrt{\det G}drd\varphi = (1 + r^2)e^{-a^2r^2}rdrd\varphi$. (One can instead calculate at first volume form in Cartesian coordinates: $dv = \sqrt{\det G}dxdy = (1 + x^2 + y^2)e^{-a^2(x^2+y^2)}dxdy$ then perform transformation to polar coordinates.)

Total area is equal to $S = \int_{\mathbf{R}^2} \sqrt{\det G}dv = \int_{\mathbf{R}^2}(1 + r^2)e^{-a^2r^2}rdrd\varphi$. Introducing new variable u: $u = a^2r^2$ we calculate this integral: $S = \int_{\mathbf{R}^2} \sqrt{\det G}dv = \int_{\mathbf{R}^2}(1 + r^2)e^{-a^2r^2}rdrd\varphi =$.

$$= 2\pi \int_0^{infty} (1 + r^2)e^{-a^2r^2})rdr = \pi \int_0^\infty \left(1 + \frac{u}{a^2}\right) e^{-u}\frac{du}{a^2} = \frac{\pi}{a^2}\left(1 + \frac{1}{a^2}\right),$$

since $\int_0^\infty e^{-u}du = \int_0^\infty ue^{-u}du = 1$. ∎

6

Consider the upper half-plane $y > 0$ with the Riemannian metric

$$G = \frac{dx^2 + dy^2}{y^2}$$

(the Lobachevsky plane).

In the Lobachevsky plane consider the domain D defined by

$$D = \{x, y: \quad x^2 + y^2 \geq 1, \ -a \leq x \leq a\},$$

where a is a parameter such that $0 < a < 1$.

Find the area of the domain D (with respect to the metric G).

Consider the points $A_t = (a, t)$, $B_t = (a, t)$ on the vertical lines delimiting the domain D. Show that the distance between these points tends to 0 if $t \to \infty$

[3 marks]

We have $G = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}$. We see that $\sqrt{\det G} = \frac{1}{y^2}$. Hence

$$S = \int_{x^2+y^2\geq 1, -a\leq x\leq a} \sqrt{\det G}dxdy = \int_{x^2+y^2\geq 1, -a\leq x\leq a} \frac{1}{y^2}dxdy = \int_{-a}^{a} \left(\int_{\sqrt{1-x^2}}^\infty \frac{dy}{y^2}\right) dx =$$

$$\int_{-a}^{a} \frac{dx}{\sqrt{1-x^2}} = 2 \arcsin a \,.$$

Consider horizontal segment which connects the points A and B. This is

$$\begin{cases} x = -a + \tau \\ y = t \end{cases} \quad 0 \leq \tau \leq 2a \,.$$

Its length of this segment is equal to

$$L = \int_0^{2a} \sqrt{\frac{x_\tau^2 + y_\tau^2}{y^2(\tau)}} d\tau = \int_0^{2a} \sqrt{\frac{1+0}{t^2(\tau)}} d\tau = \frac{2a}{t} \mathbf{t}00 \text{ if } t \to \infty$$

We se that the length of the segment tends to 0, hence the distance , the length of the shortest curve will tend to zero as well.

Remark The shortest curve which connects these points is the arc of semicircle which conneccts these points with centre at origin.

Tne answer has the following geometrical interpretation. Our domain may be considered as a isocseless triangle, with vertex at infinity. The angle between vertical line and the circle is equal to

$$\angle\alpha = \angle\beta = \frac{\pi}{2} - \text{arcsin} a \,,$$

and the sum of angles of this triangle is equal to

$$\angle\alpha + \angle\beta + \angle\gamma = \left(\frac{\pi}{2} - \arcsin a\right) + \left(\frac{\pi}{2} - \arcsin a\right) + 0 = \pi - \arcsin a.$$

Since Lobachevsky plane has cconstant curvature hence

$$S(D) = \int_D dxdy = K(\angle\alpha + \angle\beta + \angle\gamma - \pi) = 2\text{arcsin}\, a$$

We come to the answer ■

7

Let ∇ be an affine connection on the 2-dimensional manifold M such that in local coordinates (u,v), $\nabla_{\frac{\partial}{\partial u}} \left(u^2 \frac{\partial}{\partial v}\right) = 3u\frac{\partial}{\partial v} + u\frac{\partial}{\partial u}$.

Calculate the Christoffel symbols Γ^u_{uv} and Γ^v_{uv} of this connection.

[1 marks]

Using axioms for connection and definition of the Christoffel symbols we have

$$\nabla_{\frac{\partial}{\partial u}} \left(u^2 \frac{\partial}{\partial v}\right) = \nabla_{\partial_u}(u^2)\partial_v + u^2 \nabla_{\partial_u}\partial_v = 2u\partial_v + u^2 \left(\Gamma^u_{uv}\partial_u + \Gamma^v_{uv}\partial_v\right) =$$

$$u^2\Gamma^u_{uv}\partial_u + \left(2u + u^2\Gamma^v_{uv}\right)\partial_v\,.$$

Comparing with the answer

$$\nabla_{\frac{\partial}{\partial u}}\left(u^2\frac{\partial}{\partial v}\right) = 3u\frac{\partial}{\partial v} + u\frac{\partial}{\partial u}$$

we see that $\Gamma^u_{uv} = \frac{u}{u^2} = \frac{1}{u}$ and $\Gamma^v_{uv} = \frac{3u-2u}{u^2} = \frac{1}{u}$ ■

8

a) Let ∇ be an arbitrary connection on n-dimensional manifold M and let $\{\Gamma^i_{km}(x)\}$ be the Christoffel symbols of this connection. Let let $\omega = \omega_i(x)dx^i$ be a differential form on M. Show that

$$\tilde{\Gamma}^i_{km} = \Gamma^i_{km} + \delta^i_k\omega_m \qquad (8a)$$

are Christoffel symbols of the new connection.

(b) *Let $\Gamma^{i(1)}_{km}$ be the Christoffel symbols of a connection $\nabla^{(1)}$ and $\Gamma^{i(2)}_{km}$ be the Christoffel symbols of a connection $\nabla^{(2)}$.*

Show that the linear combinations $\frac{2}{3}\Gamma^{i(1)}_{km} + \frac{1}{3}\Gamma^{i(2)}_{km}$, are Christoffel symbols for some new connection.

Explain why $\frac{1}{2}\Gamma^{i(1)}_{km} + \frac{1}{3}\Gamma^{i(2)}_{km}$ are not Christoffel symbols for some connection.

[3 marks]

Γ^i_{km} are Christoffel symbols of some connection. They transform as

$$\Gamma^{i'}_{k'm'} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}} + \frac{\partial^2 x^r}{\partial x^{k'}\partial x^{m'}}\frac{\partial x^{i'}}{p x^r} \qquad (8a.1)$$

Γ^i_{km} are Christoffel symbols of some connection ∇. According Proposition (see lecture notes paragraph, 2.1.4 Space of connections, page 51 of Lecture notes)) Γ^i_{km} are Christoffel symbols of another connection $\tilde{\nabla}$ since they differ on the tensor of valency $\begin{pmatrix}1\\2\end{pmatrix}$. ■

`Another solution`

Recall that under change of coordinates x^i to $x^{i'}$ Christoffel symbols change in the following way:

$$\Gamma^{i'}_{k'm'} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\Gamma^{i'}_{k'm'} + \frac{\partial^2 x^r}{\partial x^{k'}\partial x^{m'}}\frac{\partial x^{i'}}{p x^r} \qquad (8a.1)$$

Check that symbols $\tilde{\Gamma}^i_{km}$ defined by equation (8a) indeed transform according the rule (8a.1)

$$\frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\tilde{\Gamma}^i_{km} + \frac{\partial^2 x^r}{\partial x^{k'}\partial x^{m'}}\frac{\partial x^{i'}}{\partial x^r} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\left[\Gamma^i_{km} + \delta^i_k\omega_m\right] + \frac{\partial^2 x^r}{\partial x^{k'}\partial x^{m'}}\frac{\partial x^{i'}}{\partial x^r} =$$

$$= \underbrace{\frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\Gamma^i_{km} + \frac{\partial^2 x^r}{\partial x^{k'}\partial x^{m'}}\frac{\partial x^{i'}}{\partial x^r}}_{\text{see equation (8a.1)}} + \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\delta^i_k\omega_m =$$

$$\Gamma^{i'}_{k'm'} + \frac{\partial x^{i'}}{\partial x^k}\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\omega_m = \Gamma^{i'}_{k'm'} + \frac{\partial x^{i'}}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\omega_m =$$

$$\Gamma^{i'}_{k'm'} + \delta^{i'}_{k'}\omega_{m'} = \tilde{\Gamma}^{i'}_{k'm'}\,.$$

■

`Another solution`

Denote by ∇ the conection with Christoffel symbols Γ^i_{km}. Consider new connection

$$\tilde{\nabla}\colon\ \tilde{\nabla}_{\mathbf{X}}\mathbf{Y} = \nabla_{\mathbf{X}}\mathbf{Y} + \mathbf{X}\omega(\mathbf{Y})$$

(see Example in the paragraph 2.1.4 Space of connections, page 52 of Lecture notes.)

Chirstoffel symbols of this new connection are

$$\tilde{\Gamma}^i_{km}\colon\ \tilde{\Gamma}^i_{km}\frac{\partial}{\partial x^i} = \tilde{\nabla}_{\partial_k}\partial_m = \nabla_{\partial_k}\partial_m + \frac{\partial}{\partial x^k}\omega\left(\frac{\partial}{\partial x^m}\right) = \left(\Gamma^i_{km} + \delta^i_k\omega_m\right)\cdot\frac{\partial}{\partial x^i}\,.$$

Thus we see that $\tilde{\Gamma}^i_{km} = \Gamma^i_{km} + \delta^i_k\omega_m$ are Chritoffel symbols of the new connection ■

Difference of two connections is tensor field, if ∇ is a connection, then $\tilde{\nabla}$ is another connection if and only if the difference $\tilde{\nabla} - \nabla$ is a tensor field. On the other hand one can see that if $c\nabla$ is connection, then $c = 1$. Hence $\alpha\nabla + \beta\tilde{\nabla}$ is connection if and only if

$$\alpha\nabla + \beta\tilde{\nabla} - \underbrace{\alpha(\nabla - \tilde{\nabla})}_{\text{tensor field}} = (\alpha+\beta)\tilde{\nabla}$$

is a connection, i.e. $\alpha + \beta = 1$. This explains why Show that the linear combinations $\frac{2}{3}\nabla + \frac{1}{3}\tilde{\nabla}$, is a connection and $\frac{1}{2}\nabla + \frac{1}{3}\tilde{\nabla}$ is not a connection:

$$\frac{2}{3} + \frac{1}{3} = 1\,,\quad \text{however}\ \frac{1}{2} + \frac{1}{3} \neq 1\,.$$

■