MATH 41082

Three hours

THE UNIVERSITY OF MANCHESTER

RIEMANNIAN GEOMETRY

20 May 2016
14:00—17:00

ANSWER ANY THREE OF QUESTIONS 1—4
If you answer more questions then the marks from the three best solutions will be used.
YOU HAVE TO ALSO ANSWER QUESTION 5
All questions are worth 20 marks

Electronic calculators may not be used

P.T.O.

1.

(a) Explain what is meant by saying that G is a Riemannian metric on a manifold M.

Consider the plane $\mathbf{R}^2$ with standard coordinates (x, y) equipped with Riemannian metric $G = \sigma(x, y)(dx^2 + dy^2)$.
Explain why $\sigma(x, y) > 0$.

Let $\mathbf{A}$ and $\mathbf{B}$ be two arbitrary vectors attached at some point of this plane. Explain why the cosine of the angle between these vectors does not depend on a choice of a function $\sigma(x, y)$.

In the plane with the metric G as above, consider the circle C defined by the equation $x^2 + y^2 = R^2$ and calculate its length in the case $\sigma(x, y) = e^{-x^2-y^2}$.

[8 marks]

(b) Consider the plane $\mathbf{R}^2$ with standard coordinates (x, y) equipped with Riemannian metric $G = e^{-x^2-y^2}(dx^2 + dy^2)$.

Write down the formula for the area element in this metric in coordinates (x, y) and in polar coordinates (r, φ) $(x = r\cos\varphi, y = r\sin\varphi)$.
Calculate the area of the disc, $x^2 + y^2 \leq R^2$ (in the given metric).

In the case when $R = 1$ give an example of another metric $G' = \sigma(x, y)(dx^2 + dy^2)$ such that the area of the disc $x^2 + y^2 \leq 1$ will be the same as for the metric G.

[8 marks]

(c) Explain what is meant by saying that a Riemannian manifold is locally Euclidean.

Consider in Euclidean space $\mathbf{E}^3$ the surface

$$\begin{cases} x = 3h\cos\varphi \\ y = 3h\sin\varphi \\ z = h \end{cases}, \qquad h > 0\,, \quad 0 \leq \varphi < 2\pi\,,$$

(the upper sheet of the cone).
It is known that the induced Riemannian metric on this surface is given by the formula $G = 10dh^2 + 9h^2d\varphi^2$.
Show that this surface is locally Euclidean.

[4 marks]

P.T.O.

2.

(a) Explain what is meant by an affine connection on a manifold.

Let ∇ be an affine connection on a 2-dimensional manifold M in local coordinates (u, v). It is known that $\nabla_{\frac{\partial}{\partial u}}\left(u\frac{\partial}{\partial u}\right) = \frac{\partial}{\partial u} + u\frac{\partial}{\partial v}$.
Calculate the Christoffel symbols Γ^u_{uu} and Γ^v_{uu}.

[5 marks]

(b) Explain what is meant by the induced connection on a surface in Euclidean space.

Consider a sphere in $\mathbf{E}^3$:

$$\mathbf{r}(\theta, \varphi): \quad \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases} .$$

Let ∇ be the induced connection on the sphere.

Calculate the Christoffel symbols $\Gamma^\theta_{\theta\varphi}$, $\Gamma^\varphi_{\theta\varphi}$, $\Gamma^\theta_{\varphi\theta}$ and $\Gamma^\varphi_{\varphi\theta}$.

[6 marks]

(c) Give a detailed formulation of the Levi-Civita Theorem. In particular write down the expression for the Christoffel symbols Γ^i_{km} in terms of a Riemannian metric $G = g_{ik}(x)dx^i dx^k$.

Consider the open disc $u^2 + v^2 < 1$ with the Riemannian metric

$$G = \frac{4(du^2 + dv^2)}{(1 - u^2 - v^2)^2},$$

(Poincaré disc).
Show that all Christoffel symbols of the Levi-Civita connection of this Riemannian manifold vanish at the point $u = v = 0$.

Let ∇' be a symmetric connection on the Poincaré disc such that all Christoffel symbols of this connection in coordinates (u, v) vanish identically (at all points).
Show that the connection ∇' does not preserve the metric of the Poincaré disc.
(You may wish to consider the vector field $\mathbf{A} = \frac{\partial}{\partial u}$.)

[9 marks]

P.T.O.

3.

(a) Let (M, G) be a Riemannian manifold.
Let C be a curve on M starting at the point $\mathbf{p}_1$ and ending at the point $\mathbf{p}_2$.

Explain what is meant by the parallel transport P_C along the curve C.

Explain why the parallel transport P_C is a linear orthogonal operator.

Let the points $\mathbf{p}_1$ and $\mathbf{p}_2$ coincide, so that C is a closed curve.
Let $\mathbf{a}$ be a tangent vector at the point $\mathbf{p}_1$, and $\mathbf{b} = P_C(\mathbf{a})$.
Suppose that $P_C(\mathbf{b}) = -\mathbf{a}$.
Show that vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal to each other.

[8 marks]

(b) Write down the differential equation for geodesics of a Riemannian manifold in terms of Christoffel symbols.

Explain the relation between the Lagrangian of a free particle on a Riemannian manifold and the differential equations for geodesics.

Calculate the Christoffel symbols on the Lobachevsky plane.

(You may use the Lagrangian of a free particle on the Lobachevsky plane $L = \frac{1}{2}\frac{\dot{x}^2+\dot{y}^2}{y^2}$.)

[6 marks]

(c) Explain why great circles are geodesics on a sphere in $\mathbf{E}^3$.

On the unit sphere $x^2 + y^2 + z^2 = 1$ in $\mathbf{E}^3$ consider the curve C defined by the equation $\cos\theta - \sin\theta \sin\varphi = 0$ in spherical coordinates.

Show that in the process of parallel transport along the curve C an arbitrary tangent vector to the curve remains tangent to the curve.

[6 marks]

P.T.O.

4.

(a) Let M be a surface in the Euclidean space $\mathbf{E}^3$. Let $\mathbf{e}, \mathbf{f}, \mathbf{n}$ be three vector fields defined on the points of this surface such that they form an orthonormal basis at any point, so that the vectors $\mathbf{e}, \mathbf{f}$ are tangent to the surface and the vector $\mathbf{n}$ is orthogonal to the surface. Consider the derivation formula

$$d\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\\ \mathbf{n}\end{pmatrix} = \begin{pmatrix}0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{pmatrix}\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\\ \mathbf{n}\end{pmatrix},$$

where a, b and c are 1-forms on the surface M.
Express the mean curvature and the Gaussian curvature of M in terms of these 1-forms and vector fields.
Show that $da + b \wedge c = 0$.

[6 marks]

(b) Consider the surface of a saddle in Euclidean space $\mathbf{E}^3$,

$$\mathbf{r}(u,v)\colon \quad \begin{cases} x = u \\ y = v \\ z = kuv \end{cases} . \quad k \text{ is a parameter, } k \neq 0.$$

Find vector fields $\mathbf{e}, \mathbf{f}, \mathbf{n}$ defined at the points of the saddle such that they form an orthonormal basis at any point, the vectors $\mathbf{e}, \mathbf{f}$ are tangent to the surface and the vector $\mathbf{n}$ is orthogonal to the surface.
For the obtained orthonormal basis $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ calculate the vector 1-forms $d\mathbf{e}, d\mathbf{f}, d\mathbf{n}$ and the 1-forms a, b and c at the point $\mathbf{p}$ with coordinates $u = v = 0$.
Deduce from these calculations the Gaussian curvature of the saddle at the point $\mathbf{p}$.

[8 marks]

(c) Consider a surface M in $\mathbf{E}^3$ with local coordinates (u, v) such that the induced metric of this surface is equal to $G = \sigma(u,v)(du^2 + dv^2)$.
Write down the formula expressing Gaussian curvature of this surface in terms of the function $\sigma(u, v)$. (You do not need to prove this formula.)

Let $\sigma(u,v) = \frac{1}{v^2}$. Calculate the Gaussian curvature of this surface and explain why this surface is not locally Euclidean.

[6 marks]

P.T.O.

The following question is compulsory.

5.

(a) Give a definition of curvature tensor of an affine connection.

Deduce the expresssion for the components of the curvature tensor in terms of the Christoffel symbols.

Consider 2-dimensional Riemannian manifold (M, G) with Riemannian metric
$G = e^{-ax^2-by^2}(dx^2 + dy^2)$, where a, b are parameters.

Calculate the component R_{1212} of the Riemann curvature tensor at the point $x = y = 0$ of this manifold.

[10 marks]

(b) Let $M\colon \mathbf{r} = \mathbf{r}(u, v)$ be a surface in $\mathbf{E}^3$ with induced Riemannian metric $G = \sigma(u, v)(du^2 + dv^2)$.
Using the derivation formulae deduce the expression for the Gaussian curvature of the surface M via the function $\sigma(u, v)$.

Give an example of a surface M': $\mathbf{r}' = \mathbf{r}'(u, v)$ such that the induced Riemannian metric on this surface is $G' = \lambda\sigma(u, v)(du^2 + dv^2)$, where λ is a constant ($\lambda > 0$).

What is the relation between Gaussian curvature of surfaces M' and M?

[10 marks]

END OF EXAMINATION PAPER