Two hours

THE UNIVERSITY OF MANCHESTER

RIEMANNIAN GEOMETRY

04 June 2019

09.45 – 11.45

Answer **ALL FOUR** questions in Section A (50 marks in total)
Answer **TWO** of the THREE questions in Section B (40 marks in total)
If more than TWO questions in Section B are attempted,
the credit will be given for the best TWO answers.

The use of electronic calculators is not permitted.

FEEDBACK

SECTION A

Answer **ALL** *FOUR* questions

A1.

(a) *Explain what is meant by saying that G is a Riemannian metric on a manifold M.*

(b) *Let G be a Riemannian metric on a 2-dimensional manifold M such that in local coordinates (u, v)*

$$G = ||g_{\alpha\beta}|| = \begin{pmatrix} A(u,v) & B(u,v) \\ C(u,v) & D(u,v) \end{pmatrix},$$

where $A(u,v), B(u,v), C(u,v), D(u,v)$ are smooth functions.
Show that $C(u,v) \equiv B(u,v)$ and $A(u,v) > 0$, $D(u,v) > 0$.

(c) *Show that* $\det ||g_{\alpha\beta}|| \neq 0$.

[10 marks]

Students had no problems with first and second subquestions. Just some answers on the subquestion a) was incomplete.

Some students did not give correct answer on the subquestion c). A point is that condition $\det ||g_{\alpha\beta}|| = 0$ implies that in this case the matrix has non-zero eigen vector $\mathbf{X}$ with eigenvalue 0. Hence

$$\mathbf{X}: \quad \mathbf{X} \neq 0\,, \text{and } G(\mathbf{X}, \mathbf{X}) = 0$$

This contradicts to positive definitness of the scalar product.

One student gave just `explicit` construction. The solution suggested by this student is the following. Let $||g_{\alpha\beta}|| = \begin{pmatrix} A & B \\ B & D \end{pmatrix}$ Consider vector $\mathbf{X} = \begin{pmatrix} B \\ -A \end{pmatrix}$. We have that $\mathbf{X} \neq 0$ and on the other hand

$$\langle \mathbf{X}, \mathbf{X} \rangle = (B, -A) \begin{pmatrix} A & B \\ B & D \end{pmatrix} \begin{pmatrix} B \\ -A \end{pmatrix} = A(AD - B^2)\,.$$

We see that if $\det ||g|| = 0$ then for the non-zero vector $\mathbf{X}$ the condition of positive definitness is not fullfiled. `Beautiful solution.  Is not it?`

A2.

(a) *Explain what is meant by a Riemannian manifold being locally Euclidean.*

(b) *Consider the surface of the cylinder $x^2 + y^2 = 9$ in $\mathbf{E}^3$*

$$\mathbf{r}(h, \varphi): \qquad \begin{cases} x = 3\cos\varphi \\ y = 3\sin\varphi \\ z = h \end{cases} \quad , -\infty < h < \infty\,, 0 \leq \varphi < 2\pi$$

Calculate the Riemannian metric G on this surface induced by the Euclidean metric in $\mathbf{E}^3$ in coordinates (h, φ).
Show that the surface of the cylinder is locally Euclidean by giving an example of local coordinates (u, v), which are Euclidean coordinates.

(c) *Give an example of other coordinates $(\tilde{u}, \tilde{v})$ such that these coordinates are also Euclidean coordinates.*

[10 marks]

I am happy that this question was answered good. No special problems. In particular almost all students (except few) gave an example of another local Euclidean coordinates $\tilde{u}, \tilde{v}$ The standard answer was that these new coordinates differ from initial by translation. One or two students, "rotated" initial coordinates:

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix},$$

where θ is some angle.

Just one remark on the bookwork subquestion a):

An n-dimensional Riemannian manifold (M, G) is locally Euclidean Riemannian manifold, if in a vicinity of every point $\mathbf{p}$ there exist local coordinates $u^1, \dots, u^n$ such that Riemannian metric G in these coordinates has an appearance

$$G = du^i \delta_{ik} du^k = (du^1)^2 + \cdots + (du^n)^2 .$$

Some students answering this subquestion did not emphasize that Euclidean coordinates $(u^1, \dots, u^n)$ here are `local` coordinates, they are defined in a vicinity of an arbitrary point.

A3.

(a) *Explain what is meant by an affine connection on a manifold.*

(b) *Let ∇ be an affine connection on a manifold M. Show that for arbitrary vector fields $\mathbf{X}$ and $\mathbf{Y}$ and for an arbitrary function f,*

$$e^{-f} \nabla_{\mathbf{X}} \left(e^f \mathbf{Y}\right) = (\partial_{\mathbf{X}} f)\mathbf{Y} + \nabla_{\mathbf{X}} \mathbf{Y} .$$

(c) *Explain what is meant by a canonical flat connection on Euclidean space $\mathbf{E}^n$.*
Calculate the Christoffel symbols Γ^r_{rr} and $\Gamma^r_{\varphi\varphi}$ of the canonical flat connection in the Euclidean space $\mathbf{E}^2$, where r, φ are polar coordinates ($x = r\cos\varphi$,$y = r\sin\varphi$).

[15 marks]

Students answered well the subquestions a) and b).

The subquestion c) induces some problems. They are the following:

i) defining canonical flat connection on Euclidean space it is very important to note that the Christoffel symbols vanish in `Cartesian coordinates`. It is not the full answwer to say that Christoffel symbols vanish just in `some` coordinates.

ii) It was expected the following solution for calculating the Christofel symbols for canonical connection in polar coordinates: Since canonical Christoffel symbols of canonical flat connection vanish in Cartesian coordinates, hence

$$\Gamma^r_{rr} = \frac{\partial^2 x}{\partial r^2}\frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial r^2}\frac{\partial r}{\partial y} = 0 \,, \text{since } \tfrac{\partial^2 x}{\partial r^2} = \tfrac{\partial^2 y}{\partial r^2} = 0$$

and
$$\Gamma^r_{\varphi\varphi} = \frac{\partial^2 x}{\partial \varphi^2}\frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial \varphi^2}\frac{\partial r}{\partial y} = -r\cos\varphi\frac{x}{r} - r\sin\varphi\frac{y}{r} = -r\cos^2\varphi - r\sin^2\varphi = -r \quad \blacksquare$$

Many students who calculated the Christoffel symbols $\Gamma^r rr$ and $\Gamma^r_{\varphi\varphi}$ did it in another way using Levi-Civita formula(see the discussion of the question B6 below). In the case if these calculations were right, they received the full credit [1].

A4.

(a) State the relation between the Riemannian curvature tensor of the Levi-Civita connection of a surface in $\mathbf{E}^3$ and its Gaussian curvature.

(b) Consider in $\mathbf{E}^3$ the surface of the saddle $z = kxy$ ($k \neq 0$):

$$\mathbf{r}(u,v)\colon \quad \begin{cases} x = u \\ y = v \\ z = kuv \end{cases} \quad , \quad -\infty < u, v < \infty$$

Using the shape operator calculate the Gaussian curvature at the point $\mathbf{p}$ of the saddle with coordinates $x = y = z = 0$.
Calculate the scalar curvature R of the Riemannian curvature tensor of the induced Riemannian metric on the saddle at the point $\mathbf{p}$.

(c) Explain why the saddle is not a locally Euclidean surface.

[15 marks]

This question is not difficult, and it is bookwork question, in spite of this students did not answer it in full details.

The answer on subquestion a) is the following:

Let M be a surface in $\mathbf{E}^3$, K be Gaussian curvature, $||g_{\alpha\beta}||$ induced Riemannian metric, R_{ikmn}—Riemann curvature tensor of Levi-Civita connection, R—scalar curvature of this tensor. Then

$$\frac{R}{2} = \frac{R_{1212}}{\det g} = K\,.$$

Almost all students gave only partial answer on this easy bookwork question.

The answer subquestion b) is described in detail in lecture notes, however only few students delivered detailed calculations for shape operator.

[1] in spite of the fact that it has to be proved that the answer has to be the same

SECTION B

Answer **TWO** of the three questions

B5.

(a) *Consider the stereographic projection of the sphere $x^2 + y^2 + z^2 = 1$ (without the north pole) on the plane $\mathbf{R}^2$: (u, v) are standard coordinates on $\mathbf{R}^2$ and for arbitrary point (x, y, z) on the sphere except the point $z = 1$,*

$$\begin{cases} u = \frac{x}{1-z} \\ v = \frac{y}{1-z} \end{cases}, \quad \begin{cases} x = \frac{2u}{u^2+v^2+1} \\ y = \frac{2v}{u^2+v^2+1} \\ z = \frac{u^2+v^2-1}{u^2+v^2+1} \end{cases}. \qquad (B5a)$$

The metric

$$G = \frac{4(du^2 + dv^2)}{(u^2 + v^2 + 1)^2}\,, \qquad (B5b)$$

is induced on the plane $\mathbf{R}^2$ from the metric on the sphere by the stereographic projection. Calculate the total area of $\mathbf{R}^2$ with respect to the Riemannian metric (B5b).

(b) *Let C be the intersection of the sphere with the plane $z = x$. Let C' be the image of the curve C under the stereographic projection. Show that C' is a circle.*

(c) *The curve C' divides the plane $\mathbf{R}^2$ into two domains. Find the areas of these domains of $\mathbf{R}^2$ with respect to the Riemannian metric (B5b).*

[20 marks]

This was not easy question.
You may answer subquestion a) in two different ways
a) Calculate explicitly the area of the plane with induced metric (B5 b):

$$= \int_{\mathbf{R}^2} \sqrt{\det g}\, dudv = \int_{\mathbf{R}^2} \sqrt{\left(\frac{4}{(1+u^2+v^2)^2}\right)^2}\, dudv = \int_{\mathbf{R}^2} \frac{4}{(1+u^2+v^2)^2} dudv =$$

$$\int_{\mathbf{R}^2} \frac{4}{(1+r^2)^2} r dr d\varphi = 2\pi \int_0^\infty \frac{2dz}{(1+z)^2} = -4\pi \frac{1}{1+z}\Big|_0^\infty = 4\pi, .$$

(see the lecture notes for detailes of this calculation)

b) just note that stereographic projection is isometry, hence the area of plane with induced metric (B5b) has to be equal to the area of sphere (without North pole) of radius 1, i.e. it is equal to $4\pi^2$.

Here the straightforward solution is not terribly longer than the second solution (compare with answer on subquestion c) below).

The subquestion b) was not difficult: the fact that the curve C' is a circle follows from equations (B5a):

$$z = x \Leftrightarrow \frac{u^2+v^2-1}{u^2+v^2+1} = \frac{2u}{u^2+v^2+1} \Leftrightarrow u^2+v^2-1 = 2u \Leftrightarrow (u-1)^2 + v^2 = 1$$

[2] removing a point does not change area

thus we see that C' is a circle of radius 1 with centre at the point $u = 1, v = 0$.

The subquestion c) was the most difficult subquestion of this question: just naive brute force straightforward calculations of integral

$$\int_{(u-1)^2+v^2\leq 1} \sqrt{\det G} dudv = \int_{(u-1)^2+v^2\leq 1} \frac{4dudv}{(u^2+v^2+1)^2} dudv \,.$$

are not succesfull[3].

On the other hand answer on this question can be obtained by the following beautiful and short consideration: the stereographic projection is isometry of the sphere of unit radius (without the north pole) on the plane with metric (B5a). The interior of the circle C' is the image of the halph sphere since the curve C is the great circle, and this great circle dissects the sphere on two hemispheres: Hence its area in metric (B5a) is equal to

$$\text{area of interior of } C' = \frac{\text{area of unit sphere}}{2} = \frac{4\pi}{2} = 2\pi.$$

B6.

(a) Give a detailed formulation of the Levi-Civita Theorem. In particular write down the expression for the Christoffel symbols Γ^i_{km} of the Levi-Civita connection in terms of the Riemannian metric $G = g_{ik}(x)dx^i dx^k$.

(b) Consider sphere of radius R in stereographic coordinates. It is equipped with metric

$$G = \frac{4R^4(du^2+dv^2)}{(R^2+u^2+v^2)^2} \,. \qquad (B6a)$$

Show that Christoffel symbols of Levi-Civita connection ∇ vanish at the point $u = v = 0$ in coordinates (u, v).

(c) Consider on the sphere a connection $\tilde\nabla$ such that the Riemannian metric (B6a) is invariant with respect to this connection, and the Christoffel symbols $\tilde\Gamma^u_{vv}$ and $\tilde\Gamma^v_{uu}$ of the connection $\tilde\nabla$ vanish at all points (u, v).
Show that the connection $\tilde\nabla$ does not coincide with the Levi-Civita connection ∇.
Does it contradict to Levi-Civita Theorem that the Riemannian metric (B6a) is invariant with respect to two different connections?
Justify the answer.

[20 marks]

The full answer on subquestion a) is something like this:

Let M be a Riemannian manifold with metric $G = g_{ik}dx^i dx^k$. Let ∇ be a symmetric connection on M, i.e. its Christophel symbols Γ^m_{ik} satisfies the condition: $\Gamma^m_{ik} = \Gamma^m_{ki}$. We say that symmetric connection

[3] this integral can be calculated but this needs the special technique and what is very important during exam, it needs the time!

∇ is Levi Civita connection if it preserves scalar product, i.e. if for arbitrary vectors $\mathbf{Y}, \mathbf{Z}$ at an arbitrary point

$$\partial_{\mathbf{X}} < \mathbf{Y}, \mathbf{Z} > = < \nabla_{\mathbf{X}}(\mathbf{Y}), \mathbf{Z} > + < \mathbf{Y}, \nabla_{\mathbf{X}}(\mathbf{Z}) > . \qquad (B6b)$$

Levi-Civita Theorem claims that on the Riemannian manifold (M, G) there exists uniquely defined Levi Civita connection. In local coordinates Christoffel symbols of this connection have the following appearance:

$$G = g_{ik} dx^i dx^k \Rightarrow \quad \Gamma^m_{ik}(x) = \frac{1}{2} g^{mn}(x) \left(\frac{\partial g_{in}(x)}{\partial x^k} + \frac{\partial g_{kn}(x)}{\partial x^i} - \frac{\partial g_{ik}(x)}{\partial x^n} \right) . \qquad (B6c)$$

Almost all students answered this subquestion, however many students answered it incompletely, in spite of the fact that this is bookwork question. In particular, this is very important the `uniqueness`: there is unique symmetric connection ∇ which preserves scalar product. This part of Levi-Civita Theorem is very important.

The answer on subquestion b) follows from Levi-Civita formula (B6c). Namely for Riemannian metric The derivatives of Riemannian metric vanish at the point $u = v = 0$. Indeed we have that

$$G = \sigma(du^2 + dv^2) \,, ||g_{\alpha\beta}|| = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} , \text{where } \sigma = \frac{4R^2}{(1 + u^2 + v^2)} , \qquad (B6d)$$

hence for example

$$\frac{\partial g_{uu}}{\partial v}\Big|_{u=v=0} = \frac{\partial \sigma}{\partial v}\Big|_{u=v=0} = \frac{-2 \cdot 4R^2 \cdot 2v}{(1 + u^2 + v^2)}\Big|_{u=v=0} = 0 \,.$$

Hence according to formula (B6,2) all Christoffel symbols vanish at the point $u = v = 0$ in stereographic coordinates.

Many students solved this subquestion. On the other hand some students came to the conclusion using the following "consideration": `Riemannian metric` G `is constant at the origin, hence due to equation (B6c) Christoffel symbols vanish at the origin.` This is wrong consideration! the sentence

`the metric is constant at the given point`

is senseless!

(This fact was carefully discussed during tutorials).

Now about last subquestion. It was may be the most difficult question of exam.

Denote by $\tilde{\Gamma}^u_{uu}, \Gamma^u_{uv}, \dots, \tilde{\Gamma}^v_{vv}$ Christoffel symbols of the connection $\tilde{\nabla}$. Consider preservation of scalar products (B6a) by the connection $\tilde{\nabla}$. We have that $\langle \partial_u, \partial_u \rangle = \langle \partial_v, \partial_v \rangle = \sigma$, $\langle \partial_u, \partial_v \rangle = 0$ hence due to equation (B6b)

$$0 = \partial_u \langle \partial_u, \partial_v \rangle = \langle \tilde{\nabla}_u \partial_u, \partial_v \rangle + \langle \partial_u, \tilde{\nabla}_u \partial_v \rangle = \tilde{\Gamma}^v_{uu} \langle \partial_v, \partial_v \rangle + \tilde{\Gamma}^u_{uv} \langle \partial_u, \partial_u \rangle =$$

$$\tilde{\Gamma}^v_{uu} \sigma + \tilde{\Gamma}^u_{uv} \sigma \Rightarrow \tilde{\Gamma}^u_{uv} \equiv 0 \,, \text{since } \tilde{\Gamma}^v_{uu} \equiv 0 \,,$$

and on the other hand

$$\partial_v \sigma = \partial_v \langle \partial_u, \partial_u \rangle = 2\tilde{\Gamma}^u_{vu} \sigma \Rightarrow \tilde{\Gamma}^u_{vu} = \frac{1}{2} \partial_v \log \sigma \not\equiv 0 \,.$$

i.e. connection $\tilde{\nabla}$ is not symmetric: $0 \equiv \tilde{\Gamma}^u_{uv} \neq \tilde{\Gamma}^u_{vu}$.

We see that connection ∇ is not symmetric, hence it does not coincide with symmetric Levi-Civita connection ∇.

The existence of the *non-symmetric* connection which preserves scalar metric does not contradict to Levi-Civita theorem, which says about uniqueness of *symmetirc connection* which preserves the scalar metric.

Compare it with non-symmetric connection in $\mathbf{E}^3$ which preserves scalar product (see the question 7 in Cousework).

B7.

(a) Let (M, G) be a Riemannian manifold, let C be a curve on M starting at the point $\mathbf{p}_1$ and ending at the point $\mathbf{p}_2$.
Explain what is meant by parallel transport $P_{C_{\mathbf{p}_1\mathbf{p}_2}}$ along the curve C.

(b) Show that in the case that $\mathbf{p}_1 = \mathbf{p}_2 = \mathbf{p}$ (C is a closed curve), then $P_{C_{\mathbf{p}_1\mathbf{p}_2}} = P_{C_\mathbf{p}}$ is an orthogonal operator in the tangent space $T_\mathbf{p}M$.

(c) Let M be the sphere $x^2 + y^2 + z^2 = R^2$ in $\mathbf{E}^3$ with induced Riemannian metric, and let the curve C on M be an intersection of the sphere with a plane $x + z = R$. Take the point $\mathbf{p} = (R, 0, 0)$ on the curve C and consider vectors $\mathbf{Y} = \frac{\partial}{\partial y}$ and $\mathbf{Z} = \frac{\partial}{\partial z}$ which are tangent to the sphere at the point $\mathbf{p}$. Find the result of the parallel transport of these vectors along the closed curve C.

[20 marks]

The last question was not easy, however it was in some sence 'predictable'. The subquestions a) b) were carefully analyzed in lecture notes.

To answer subquestion c) you have to note that the plane $x + z = R$ dissects from the sphere the segment. Area of the segment is equal to

$$S = 2\pi Rh\,, \qquad (B7a)$$

where h is the height of the segment. One can see that the distance from origin till plane $z + x = R$ is equal to $\frac{R\sqrt{2}}{2}$, hence the height of the segment is equal to $h = R - \frac{R\sqrt{2}}{2}$ and area of the segment is equal to

$$S = 2\pi Rh = 2\pi R^2 \left(1 - \frac{\sqrt{2}}{2}\right)\,. \qquad (B7b)$$

(see for details the last formula in subsection 1.5.2. in Lecture notes)

According to the Theorem on parallel transport every vector rotates on the angle

$$\angle\Phi = \int_D K d\sigma = \frac{1}{R^2} S(D) = \frac{1}{R^2} 2\pi R^2 \left(1 - \frac{\sqrt{2}}{2}\right) = \pi(2 - \sqrt{2})\,. \qquad (B7c)$$

If one calculates the angle of rotation, then one can easy deduce that the result of paralel transport is

$$P_C \begin{pmatrix} \mathbf{Y} \\ \mathbf{Z} \end{pmatrix} = \begin{pmatrix} \cos\Phi & -\sin\Phi \\ \sin\Phi & \cos\Phi \end{pmatrix} \begin{pmatrix} \mathbf{Y} \\ \mathbf{Z} \end{pmatrix}\,,$$

where Φ is given by equation (B7c).

This question was not answered succesfully in spite of the fact that tit is not very difficult. Paradoxially maybe the most difficult part was to use the formula (B7a) [4] which belongs to elementary mathematics.

[4] it is not worthless to note that many years ago the formula (B7a) was just the part of school mathematics

END OF EXAMINATION PAPER