Homework 0

1 Define on the circle $x^2 + y^2 = R^2$ the following four local coordinates:
a) angle coordinate which is not define at the point $(R, 0)$
b) angle coordinate which is not define at the point $(-R, 0)$
c) stereographic coordinate with respect to North pole
d) stereographic coordinate with respect to South pole
Find transition functions between these coordinates

2 Define on the sphere $x^2 + y^2 + z^2 = a^2$ the following local coordinates:
a) spherical coordinates,
b) stereographic coordinates with respect to North pole
c) stereographic coordinates with respect to South pole
Find transition functions between these coordinates

3 Consider 2-dimensional manifold, the domain $D = \mathbf{E}^2 \backslash \mathbf{0}$ in $\mathbf{E}^2$ (plane without a point). Let (x, y) be standard Cartesian coordinates in this domain. Consider new local coordinates (u, v) such that

$$u = \frac{x}{x^2 + y^2}\,, \quad v = \frac{y}{x^2 + y^2}\,, \quad (x \neq 0, y \neq 0) \quad \text{(inversion)}\,. \tag{1}$$

Calculate Jacobian matrix of changing of local coordinates, and its determinant.

Write down inverse transition functions $x = x(u, v)$, $y = y(u, v)$ from coordinates (u, v) to coordinates (x, y).

† Answer questions above using holomorphic and antiholomorphic functions

4 Consider 3-dimensional manifold, the domain $D = \mathbf{E}^3 \backslash \mathbf{0}$ in $\mathbf{E}^3$ (space without a point). Let (x, y, z) be standard Cartesian coordinates in this domain. Consider new local coordinates (u, v, w) such that

$$u = \frac{x}{x^2 + y^2 + z^2}\,, \; v = \frac{y}{x^2 + y^2 + z^2}\,, \; w = \frac{z}{x^2 + y^2 + z^2}\,, \quad (x \neq 0, y \neq 0, z \neq 0) \quad \text{(inversion)}\,.$$

Write down inverse transition functions from coordinates (u, v, w) to coordinates (x, y, z).

Calculate Jacobian matrix of changing of local coordinates, and its determinant.

Hint: May be it is easier to calculate the determinant performing calculations in general case when dimension n is an arbitrary number

5 Consider the following tensor fields given in local coordinates x^i by equations

$$\mathbf{A} = x^i \frac{\partial}{\partial x^i}\,, \quad G = g_{ik}(x) dx^i \otimes dx^k\,, \quad Q = F^{ik}(x) \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^k}\,.$$

Express these tensor fields in new local coordinates $y^i = 2x^i$.

6 Consider in manifold $\mathbf{E}^2 \backslash \mathbf{0}$ (see Exercise 3, above) vector field $\mathbf{K} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ and differential 1-form $\omega = \frac{x dy - y dx}{x^2 + y^2}$.

Express these objects in inversed local coordinates (u, v) (see equation (1)).