Homework 2

1 Consider an upper half-plane $(y > 0)$ in $\mathbf{R}^2$ equipped with Riemannian metric

$$G = \sigma(x,y)(dx^2 + dy^2)\,, \qquad (1)$$

a) Show that $\sigma > 0$,

Consider two vectors $\mathbf{A} = 2\partial_x$ and $\mathbf{B} = 12\partial_x + 5\partial_y$ attached at the point $(x,y) = (1,2)$,

b) calculate the cosine of the angle between these vectors, and show that the answer does not depend on the choice of the function $\sigma(x,y)$.

c) Calculate the lengths of these vectors in the case if

$$\sigma = \frac{1}{y^2}\,, \qquad \text{(hyperbolic (Lobachevsky) metric)} \qquad (2)\,,$$

d) Calculate the length of the segments $\begin{cases} x = a+t \\ y = b \end{cases}$ and and $\begin{cases} x = a \\ y = b+t \end{cases}$, $0 \leq t \leq 1$ for Lobachevsky plane (i.e. if condition (2) is obeyed)

e) Consider two curves L_1 and L_2 in upper half-plane with metric given by equation (1) such that

$$L_1 = \begin{cases} x = f(t) \\ y = g(t) \end{cases}, \quad \text{and } L_2 \begin{cases} x = g(t) \\ y = f(t) \end{cases}, \qquad 0 \leq t \leq 1\,,$$

where $f(t), g(t)$ are arbitrary functions $(f(t) > 0, g(t) > 0)$.

Show that these curves have the same length in the case if $\sigma(x,y) = \frac{1}{(1+x^2+y^2)^2}$.

2 Let (M,G) be 2-dimensional Riemannian manifold with Riemannian metric G such that in local coordinates (u,v) it has apperance

$$G = A(u,v)du^2 + 2B(u,v)dudv + C(u,v)dv^2\,, ||g_{ik}|| = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

Consider vector fields $\mathbf{A} = t\frac{\partial}{\partial u} + r\frac{\partial}{\partial v}$ and $\mathbf{B} = r\frac{\partial}{\partial u} - t\frac{\partial}{\partial v}$ where t, r are arbitrary coefficients.

a) Calculate the scalar product $\langle \mathbf{A}, \mathbf{B} \rangle_G$ in the case if u, v are conformal coordinates.

b) Show that condition

$$\langle \mathbf{A}, \mathbf{B} \rangle_G = 0\,, \quad \text{for arbitrary } t, r \in \mathbf{R}$$

implies that u, v are conformal coordinates.

3 Write down the standard Euclidean metric on $\mathbf{E}^2$ in polar coordinates

4 Consider the Riemannian metric on the circle of the radius R induced by the Euclidean metric on the ambient plane.

a) Express it using polar angle as a coordinate on the circle.

b) Express the same metric using stereographic coordinate obtained by stereographic projection of the circle on the line, passing through its centre.

5 Consider the Riemannian metric on the sphere of the radius R induced by the Euclidean metric on the ambient 3-dimensional space.

a) Express it using spherical coordinates on the sphere.

b) Express the same metric using stereographic coordinates u, v obtained by stereographic projection of the sphere on the plane, passing through its centre.

6 a) Let (u, v) be local coordinates on 2-dimensional Riemannian manifold (M, G) such that Riemannian metric has an appearance $G = du^2 + u^2 dv^2$ in these coordinates. Show that there exist local coordinates x, y such that $G = dx^2 + dy^2$.

b) Let (u, v) be local coordinates on 2-dimensional Riemannian manifold (M, G) such that Riemannian metric has an appearance $G = du^2 + \sin^2 u dv^2$ in these coordinates.

Do there exist coordinates x, y such that $G = dx^2 + dy^2$?