Homework 3

1 Consider the saddle $z - xy = 0$ in $\mathbf{E}^3$:

$$\begin{cases} x = u \\ y = v \\ z = uv \end{cases},$$

a) Calculate the induced Riemannian metric on the saddle

b) Show that for every point $\mathbf{p}$ on the saddle, there exist two straight lines in $\mathbf{E}^3$ which pass trhough this point and belong to the saddle

2 Let S^2 be a sphere of unit length in $\mathbf{E}^3$.

a) Introduce on the sphere spherical coordinates θ, φ, and calculate induced Riemannian metric on the sphere in these coordinates.

b) Consider on sphere stereographic coordinates and show that these coordinates are conformal coordinates.

c) Consider on the sphere local coordinates t, r such that $\begin{cases} t = \log \tan \frac{\theta}{2} \\ r = \varphi \end{cases}$, where θ, φ are spherical coordinates. Show that these coordinates are also conformal coordinates on the sphere.

c) compare these coordinates with coordinates with stereographic coordinates

3 Consider cone $x^2 + y^2 - k^2 z^2 = 0$ in $\mathbf{E}^3$:

$$\mathbf{r} = \mathbf{r}(h, \varphi)\colon \quad \begin{cases} x = kh \cos \varphi \\ y = kh \sin \varphi \\ z = h \end{cases},$$

(Strickly speaking we consider the cone without apex, the point $x = y = z = 0$.)

a) Calculate induced Riemannian metric on the surface of the cone in coordinates h, φ

b) Consider on the surface of the cone local coordinates t, r such that $\begin{cases} t = \log h \\ r = c\varphi \end{cases}$, where c is a parameter ($c \neq 0$)

Calculate induced Riemannian metric in these coordinates, and find a value of parameter c such that coordinates t, r become conformal coordinates on C

4 Consider plane $\mathbf{R}^2$ with Riemannian metric given in Cartesian coordinates (x, y) by the formula

$$G_{\mathbf{R}^2} = \frac{a\left((dx)^2 + (dy)^2\right)}{(1 + x^2 + y^2)^2}, \quad (a > 0), \tag{1}$$

and a sphere S_r $x^2 + y^2 + z^2 = r^2$ (of the radius r) in the Euclidean space $\mathbf{E}^3$.

Consider the following map F from the plane $\mathbf{R}^2$ to the sphere

$$F(x, y)\colon \begin{cases} u = rx \\ v = ry \end{cases},$$

where (u, v) are stereographic coordinates of the sphere ($u = \frac{rx}{r-z}$, $v = \frac{ry}{r-z}$).

The map F is a diffeomorphism of $\mathbf{R}^2$ on the sphere without North pole (the point N with coordinates $x = 0, y = 0, z = r$), F: $\mathbf{R}^2 \to S_r \backslash N$

a) Write down the Riemannian metric G_S on the sphere in stereographic coordinates.

b) Write down the metric on the plane $\mathbf{R}^2$, induced by the diffeomorphism F (the pull-back $F^* G_S$ of the metric on the sphere).

c) Find parameter a such that F is isometry of the plane $\mathbf{R}^2$ equipped with Riemannian metric (1) and $S_r \backslash N$, i.e. $G_{\mathbf{R}^2} = F^* G_{S_r}$

5 Consider Lobachevsky (hyperbolic) plane: an upper half-plain ($y > 0$) in $\mathbf{R}^2$ equipped with Riemannian metric

$$G = \frac{dx^2 + dy^2}{y^2},$$

a) Show that the map $\begin{cases} x = \lambda x' \\ y = \lambda y' \end{cases}$, $(\lambda > 0)$ is an isometry of the Lobachevsky plane on itself

b) Give an example of another isometry of Lobachevsky plane

6 a) Show that surface of the cylinder $x^2 + y^2 = a^2$ in $\mathbf{E}^3$ is locally Euclidean Riemannian surface, (is locally isometric to Euclidean plane).

b) In exercise 3 we found conformal coordinates on the surface of cone. In fact one cah show that surface of the cone $\begin{cases} x^2 + y^2 - k^2 z^2 = 0 \\ z \neq 0 \end{cases}$ in $\mathbf{E}^3$ is locally Euclidean Riemannian surface, (is locally isometric to Euclidean plane).

(To do this consider on the surface of the cone local coordinates u, v such that $\begin{cases} u = \alpha r \cos \beta\varphi \\ v = \alpha r \sin \beta\varphi \end{cases}$, *where α, β are parameter $(\alpha, \beta \neq 0)$. Calculating induced Riemannian metric in these coordinates we will find values of parameters α, β such that coordinates u, v become locally Euclidean coordinates on C.)*

c) Find a relation of local Eucldean coordinates (u, v) on the cone with coordinates t, r considered in question 3)