Homework 4

1 Consider parallelogram $\Pi_{\mathbf{a},\mathbf{b}}$ formed by two vectors in Euclidean space $\mathbf{E}^2$:

$$\Pi_{\mathbf{a},\mathbf{b}} = u\mathbf{a} + v\mathbf{b}\,, \quad 0 \le u \le 1, 0 \le v \le 1\,,$$

$$\mathbf{r}(u,v) = \mathbf{a}u + \mathbf{b}v = \begin{pmatrix} a_x \\ a_y \end{pmatrix} u \begin{pmatrix} b_x \\ b_y \end{pmatrix} v = \begin{cases} x = a_x u + b_x v \\ y = a_y u + b_y v \end{cases}.$$

a) Write down standard Euclidean metric $G = dx^2 + dy^2$ in coordinates (u, v).

b) Calculate the area of parallelogram $\Pi_{\mathbf{a},\mathbf{b}}$ using Riemannian volume form

c) Compare the answer with stadard formula for area of parallelogram (*See subsection 1.5.1 "Motiation. Gramm formula for volume of parallellepiped"*)

2 a) Consider the domain D on the cone $x^2 + y^2 - k^2 z^2$ defined by the condition $0 < z < H$. Find an area of this domain using induced Riemannian metric. Compare with the answer when using standard formulae.

3 Find an area of the segment of the height h of the sphere of radius R (surface: $x^2 + y^2 + z^2 = R^2, \le a \le a + h$ for an arbitrary a: $-R \le a \le R - h$)

4 Find an area of 2-dimensional sphere of radius R using explicit formulae for induced Riemannian metric in stereographic coordinates.

5 Show that two spheres of different radii in Euclidean space are not isometric to each other, i.e. there is no an isometry of one sphere on another.

6 In exercise 4 of previous homework you have considered Riemannian manifolds $(\mathbf{R}^2, G_{(1)})$ and $(\mathbf{R}^2, G_{(2)})$, where

$$G_{(1)} = \frac{a(dx^2 + dy^2)}{(1 + x^2 + y^2)^2}\,, \quad \text{and} \quad G_{(2)} = \frac{4R^4(du^2 + dv^2)}{(R^2 + u^2 + v^2)^2}$$

(The second manifold is sphere of radius R without North pole in stereographic coordinates) You proved in the previous homework that in the case if $a = 4R^2$ then under isometry $\begin{cases} u = Rx \\ v = Ry \end{cases}$ these Riemannina manifolds are isometric.

a) Calculae the total are of the plane $\mathbf{R}^2$ for these both Riemannian manifolds.

b) On the base of the calculations above, and using the result of previous exercise, prove now more strong statement, that in the case if the condition $a = 4R^2$ is not obeyed, then these manifolds *are not isometric.*

7 Let D be a domain in Lobachevsky plane which is lying between lines $x = a, x = -a$ and outside of the disc $x^2 + y^2 = 1$, $(0 < a < 1)$: $D = \{(x, y)\colon\ |x| < a, x^2 + y^2 > 1\}$,

a) Find the area of this domain.

b) Find the angles between lines and arc of the circle.

Lobachevsky plane, i.e. hyperbolic plane is the upper half plane with Riemannian metric $\frac{dx^2+dy^2}{y^2}$ *in Cartesian coordinates* x, y $(y > 0)$.

8* Find a volume of n-dimensional sphere of radius a. (You may use Riemannian metric in stereographic coordinates, or you may do it in other way... You just have to calculate the answer.)

Hint: One way to do it is the following. Denote by σ_n *the volume of* n*-dimensional unit sphere embedded in Euclidean space* $\mathbf{E}^{n+1}$. *One can see that the volume of* n*-dimensional sphere of the radius* R *equals to* $\sigma_n R^n$. *We need to calculate just* σ_n. *Consider the following integral:* $I_k = \int e^{-r^2} dx^1 dx^2 \dots dx^k$, *where* $r^2 = (x^1)^2 + (x^2)^2 + \dots + (x^k)^2$. *One can see that on one hand* $I_k = \int e^{-r^2} dx^1 dx^2 \dots dx^k = \left(\int e^{-x^2} dx\right)^n = \pi^{\frac{n}{2}}$, *and on the other hand* $I_k = \int e^{-r^2} dx^1 dx^2 \dots dx^k = \sigma_{k-1} \int e^{-r^2} r^{k-1} dr$. *Comparing these integrals we calculate* σ_n.