Homework 6

1. Calculate Levi-Civita connection of the metric $G = a(u,v)du^2 + b(u,v)dv^2$

a) in the case if functions $a(u,v)$, $b(u,v)$ are constants.

b) in the case if u, v are conformal coordinates

c) in the general case

2 Calculate Levi-Civita connection of the Riemannian metric $G = e^{-x^2-y^2}(dx^2+dy^2)$ at the point $x = y = 0$.

3. Calculate Levi-Civita connection of Euclidean plane in polar coordinates

4 Calculate Levi-Civita connection of the Riemannian metric induced on cylinder $x^2 + y^2 = a^2$ in coordinates h, φ:

$$\mathbf{r}(h,\varphi)\colon \begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = h \end{cases} .$$

5. Calculate Levi-Civita connection of the Riemannian metric induced on the cone $x^2 + y^2 - k^2 z^2 = 0$ in coordinates h, φ

$$\mathbf{r}(h,\varphi)\colon \begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases} .$$

Do there exist coordinates on the cone such that Christoffel symbols of Levi-Civita connection of induced metric vanish in these coordinates?

6. Calculate Levi-Civita connection of the metric $G = R^2(d\theta^2 + \sin^2\theta d\varphi^2)$ on the sphere.

7 Let $\mathbf{E}^2$ be the Euclidean plane with the standard Euclidean metric $G_{\text{Eucl.}} = dx^2+dy^2$.

You know that for the Levi-Civita connection of this metric the Christoffel symbols vanish in the Cartesian coordinates x, y. (Why?)

Let ∇ be a symmetric connection on the Euclidean plane $\mathbf{E}^2$ such that its Christoffel symbols satisfy the condition $\Gamma^y_{xy} = \Gamma^y_{yx} \neq 0$.

Show that for vector fields $\mathbf{A} = \partial_x$ and $\mathbf{B} = \partial_y$, $\partial_{\mathbf{A}} \langle \mathbf{B}, \mathbf{B} \rangle \neq 2\langle \nabla_{\mathbf{A}} \mathbf{B}, \mathbf{B} \rangle$, i.e. the connection ∇ does not preserve the Euclidean scalar product $\langle\ \ ,\ \ \rangle$.

Here is described the quickest way to calculate Levi-Civita connection

8 † Consider the Lagrangian of a free particle $L = \frac{1}{2} g_{ik}\dot{x}^i\dot{x}^k$ for Riemannian manifold with a metric $G = g_{ik}dx^i dx^k$. Write down the Euler-Lagrange equations of motion for this Lagrangian and compare them with differential equations for geodesics on this Riemannian manifold. In fact show that

$$\underbrace{\frac{\partial L}{\partial x^i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}}_{\text{Euler-Lagrange equations}} \qquad \Leftrightarrow \qquad \underbrace{\frac{d^2x^i}{dt^2} + \Gamma^i_{km}\dot{x}^k\dot{x}^m = 0}_{\text{Equations for geodesics}} \ , \tag{1}$$

where

$$\Gamma^i_{km} = \frac{1}{2} g^{ij} \left(\frac{\partial g_{jk}}{\partial x^m} + \frac{\partial g_{jm}}{\partial x^k} - \frac{\partial g_{km}}{\partial x^j} \right) . \tag{2}$$

Write down the Lagrangian of a free particle $L = \frac{1}{2} g_{ik} \dot{x}^i \dot{x}^k$ and using the Euler-Lagrange equations for this Lagrangian calculate the Christoffel symbols (the Christoffel symbols of the Levi-Civita connection) for

a) Euclidean plane in polar coordinates

b) for the sphere of radius R

c) for the Lobachevsky plane

Compare with the results that you obtained using another methods.