Homework 9

1 On the sphere $x^2+y^2+z^2=R^2$ in $\mathbf{E}^3$ consider a circle C which is the intersection of the sphere with the plane $z=R-h$, $0<h<R$

Let $\mathbf{X}$ be an arbitrary vector tangent to the sphere at a point of C.

Find the angle between $\mathbf{X}$ and the result of parallel transport of $\mathbf{X}$ along C.

2 Write down components of the curvature tensor R^i_{kmn} in terms of the Christoffel symbols Γ^i_{km} and its derivatives.

3 *For every of the statements below prove it or show that it is wrong considering counterexample.*

a) If there exist coordinates u,v such that Riemannian metric G at the given point $\mathbf{p}$ is equal to $G=du^2+dv^2$ in these coordinates, then the Riemann curvature tensor at the point $\mathbf{p}$ vanishes.

b) If all first derivatives of components of Riemannian metric in coordinates u,v vanish at the given point $\mathbf{p}$:

$$\frac{\partial g_{ik}(u,v)}{\partial u}\Big|_{\mathbf{p}}=\frac{\partial g_{ik}(u,v)}{\partial v}\Big|_{\mathbf{p}}=0\,,$$

then the Riemann curvature tensor also vanishes at this point.

c) If all first and second derivatives of components of Riemannian metric

$$\frac{\partial g_{ik}(u,v)}{\partial u}\Big|_{\mathbf{p}}=\frac{\partial g_{ik}(u,v)}{\partial v}\Big|_{\mathbf{p}}=\frac{\partial^2 g_{ik}(u,v)}{\partial u^2}\Big|_{\mathbf{p}}=\frac{\partial^2 g_{ik}(u,v)}{\partial u\partial v}\Big|_{\mathbf{p}}=\frac{\partial^2 g_{ik}(u,v)}{\partial v^2}\Big|_{\mathbf{p}}=0$$

vanish at the given point then the Riemann curvature tensor also vanishes at this point.

4 Let $x^i, i=1,\dots,n$ be local coordinates on Riemannian manifold M such that for Riemannian metric tensor $G=g_{ik}(x)dx^idx^k$ the following condition holds: first derivatives of all components of metric tensor vanish at the given point $\mathbf{p}$:

$$\frac{\partial g_{ik}(x)}{\partial x^m}\Big|_{\mathbf{p}}=0\ \ (i,k,m=1,\dots,n)\,. \tag{4}$$

Write down components R^i_{kmn} of the Riemann curvture tensor in terms of Christoffel symbols Γ^i_{km} and its derivatives at the point $\mathbf{p}$ in these local coordinates

Find points on the sphere of radius a in $\mathbf{E}^3$ such that condition (4) holds in spherical coordinates, and calculate Riemann curvature tensor in these points of sphere

Calculate Riemann scalar curvature at arbitrary point of the sphere.

Compare results of calculations with formula for relation between the Gaussian curvature and Riemann curvature tensor for surfaces in $\mathbf{E}^3$:

$$K=\frac{R}{2}=\frac{R_{1212}}{\det g}\,.$$