Solutions 3

1 *Consider the saddle $z - xy = 0$ in $\mathbf{E}^3$:*

$$\begin{cases} x = u \\ y = v \\ z = uv \end{cases},$$

a) Calculate the induced Riemannian metric on the saddle

b) Show that for every point $\mathbf{p}$ on the saddle, there exist two straight lines which pass trhough this point and belong to the saddle

a) We have

$$dx = du, dy = dv, dz = udv + vdu\,,$$

$$G_{\text{saddle}} = dx^2 + dy^2 + dz^2\big|_{\mathbf{r}=\mathbf{r}(u,v)} = du^2 + dv^2 + (udv + vdu)^2 =$$

$$(1+v^2)du^2 + 2uvdudv + (1+u^2)dv^2\,, \quad ||g_{\alpha\beta}|| = \begin{pmatrix} 1+v^2 & uv \\ uv & 1+u^2 \end{pmatrix}.$$

b) Let $\mathbf{p} = (x_0, y_0, z_0) = (u_0, v_0, u_0 v_0)$ be an arbitrary point on the saddle. Consider the lines

$$I\colon \begin{cases} x(t) = x_0 + t = u_0 + t \\ y(t) = y_0 = v_0 \\ z(t) = z_0 + v_0 t = u_0 v_0 + v_0 t \end{cases} \qquad I\colon \begin{cases} x(\tau) = x_0 = u_0 \\ y(\tau) = y_0 + \tau = v_0 + \tau \\ z(\tau) = z_0 + u_0\tau = u_0 v_0 + u_0 \tau \end{cases}$$

These both lines pass through the point $\mathbf{p}$ (for $t = \tau = 0$) and they belong to the saddle, since $x(t)y(t) = z(t)$ and $x(\tau)y(\tau) = z(\tau)$.

2 *Let S^2 be a sphere of unit length in $\mathbf{E}^3$.*

a) Introduce on the sphere spherical coordinates θ, φ, and calculate induced Riemannian metric on the sphere in these coordinates

b) Consider on the sphere local coordinates t, r such that $\begin{cases} t = \log\tan\frac{\theta}{2} \\ r = \varphi \end{cases}$. Show that these coordinates are conformal coordinates

c) Consider coordinates on the sphere which are stereographic coordinates with respect to South pole of the sphere.

Show that these coordinates are also conformal coordinates and compare these coordinates with coordinates t, r considered above

We have $\begin{cases} x = \sin\theta\cos\varphi \\ y = \sin\theta\sin\varphi \\ z = \cos\theta \end{cases}$ Induced Riemannian metric is equal to $G_{S^2} = d\theta^2 + \sin^2\theta d\varphi^2$ (see lecture notes)

For new coordinates t, r

$$\begin{cases} t = \log\tan\frac{\theta}{2} \\ r = \varphi \end{cases} \Leftrightarrow \begin{cases} \theta = 2\mathrm{arctan}\, e^t \\ \varphi = r \end{cases} \tag{1}$$

and

$$\sin^2\theta = 4\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2} = 4\cdot\frac{e^{2t}}{1+e^{2t}}\cdot\frac{1}{1+e^{2t}}\,,$$

hence

$$d\theta^2 + \sin^2\theta d\varphi^2 = \left(\frac{2e^t dt}{1+e^{2t}}\right)^2 + \frac{4e^{2t}}{(1+e^{2t})^2}dr^2 = \frac{4e^{2t}}{(1+e^{2t})^2}(dt^2 + dr^2)\,.$$

Thus these coordinates are conformal

b) Stereographic coordinates u, v with respect to South pole obey condition the following condition: if a point (x, y, z) is on the sphere (we suppose that origin is at the centre) then

$$\frac{u}{x} = \frac{v}{y} = \frac{1}{1+z}\,,$$

thus

$$u = \frac{x}{1+z} = \frac{\sin\theta\cos\varphi}{1+\cos\theta} = \tan\frac{\theta}{2}\cos\varphi\,,\quad v = \frac{y}{1+z} = \frac{\sin\theta\sin\varphi}{1+\cos\theta} = \tan\frac{\theta}{2}\sin\varphi\,. \tag{2}$$

Stereographic coordinates are also conformal coordinates: in these coordinates Riemannian metric is

$$G = \frac{4(du^2 + dv^2)}{(1+u^2+v^2)^2}$$

(see lecture notes). Now analyze relations between stereographic conformal coordinates, and conformal coordinates t, r which we analysed above (see equation (1)).

One can see that due to equation (2) these conformal coordinates are related with coordinates t, r by the following function:

$$w = u + iv = \tan\frac{\theta}{2}\cos\varphi + i\tan\frac{\theta}{2}\sin\varphi\tan\frac{\theta}{2}e^{i\varphi} = e^{t+ir}\,,\quad \text{i.e.} \log w = t + ir\,.$$

(Here log is a complex logarithm.)

3 *Consider cone* $x^2 + y^2 - k^2z^2 = 0$ *in* $\mathbf{E}^3$*:*

$$\mathbf{r} = \mathbf{r}(h, \varphi)\colon\quad \begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases},$$

(Strickly speaking we consider the cone without apex, the point $x = y = z = 0$*.)*

a) Calculate induced Riemannian metric on the surface of the cone

b) Consider on the surface of the cone local coordinates t, r such that $\begin{cases} t = \log h \\ r = c\varphi \end{cases}$, *where c is a parameter ($c \neq 0$)*

Calculate induced Rimeannian metric in these coordinates, and find a value of parameter c such that coordinates t, r become conformal coordinates on C

For induced metric on cone

$$G_{\text{cone}} = \left(d\left(kh\cos\varphi\right)\right)^2 + \left(d\left(kh\sin\varphi\right)\right)^2 + (dh)^2 =$$

$$\left(-kh\sin\varphi d\varphi + k\cos\varphi dh\right)^2 + (kh\cos\varphi d\varphi + k\sin\varphi dh) + (dh)^2 = (1+k^2)dh^2 + k^2h^2d\varphi^2 =$$

(you may see also lecture notes or beginning of the solution of exercise **5** below).

We have

$$\begin{cases} t = \log h \\ r = c\varphi \end{cases} \Leftrightarrow \begin{cases} h = e^t \\ \varphi = \frac{r}{c} \end{cases},$$

and

$$G_{\text{cone}} = (1+k^2)dh^2 + k^2h^2d\varphi^2 = (1+k^2)e^{2t}dt^2 + \frac{k^2e^{2t}}{c^2}dr^2 \,.$$

These coordinates become conformal if

$$(1+k^2)e^{2t} = \frac{k^2e^{2t}}{c^2} \Leftrightarrow c = \frac{\pm k}{\sqrt{1+k^2}} \,.$$

4 *Consider plane $\mathbf{R}^2$ with Riemannian metric given in Cartesian coordinates (x, y) by the formula*

$$G_{\mathbf{R}^2} = \frac{a\left((dx)^2 + (dy)^2\right)}{(1+x^2+y^2)^2}\,, \quad (a > 0)\,, \tag{3}$$

and a sphere S_r $x^2+y^2+z^2 = r^2$ (of the radius r) in the Euclidean space $\mathbf{E}^3$.

Consider the following map F from the plane $\mathbf{R}^2$ to the sphere

$$F(x,y)\colon \begin{cases} u = rx \\ v = ry \end{cases}, \tag{4}$$

where (u, v) are stereographic coordinates of the sphere ($u = \frac{rx}{r-z}$, $v = \frac{ry}{r-z}$).

The map F is a diffeomorphism of $\mathbf{R}^2$ on the sphere without North pole (the point N with coordinates $x = 0, y = 0, z = r$), $F\colon\ \mathbf{R}^2 \to S_r \backslash N$

a) Write down the Riemannian metric G_S on the sphere in stereographic coordinates.

b) Write down the metric on the plane $\mathbf{R}^2$, induced by the diffeomorphism F (the pull-back F^*G_S of the metric on the sphere).

c) Find parameter a such that F is isometry of the plane $\mathbf{R}^2$ equipped with Riemannian metric (1) and $S_r \backslash N$, i.e. $G_{\mathbf{R}^2} = F^* G_{S_r}$

a) The Riemannian metric on the sphere in stereographic coordinates is

$$G_{S_r} = \frac{4r^4(du^2 + dv^2)}{(r^2 + u^2 + v^2)^2},$$

where r is a radius of sphere (see lecture notes or Homework 2).

b)

$$F^*\left(G_{S_r}\right) = F^*\left(\frac{4r^4(du^2 + dv^2)}{(r^2 + u^2 + v^2)^2}\right) = \left(\frac{4r^4(du^2 + dv^2)}{(r^2 + u^2 + v^2)^2}\right)\Big|_{u=rx, v=ry} =$$

$$\left(\frac{4r^4(r^2dx^2 + r^2dy^2)}{(r^2 + r^2x^2 + r^2y^2)^2}\right) = \frac{4r^2(dx^2 + dy^2)}{(1 + x^2 + y^2)^2} \tag{5}$$

c) First note that map $F\colon \mathbf{R}^2 \leftrightarrow S_r \backslash N$ (see equation (4)) is diffeomorphism.

Comparing equation (5) for Riemannina metric on the sphere (without North pole, in coordinates x, y) with Riemannian metric (3) on the plane we see that the diffeomorphism $F\colon \mathbf{R}^2 \leftrightarrow S_r \backslash N$ is an isometry if $a = 4r^2$.

5 *Consider Lobachevsky (hyperbolic) plane: an upper half-plain ($y > 0$) in $\mathbf{R}^2$ equipped with Riemannian metric*

$$G = \frac{dx^2 + dy^2}{y^2},$$

a) Show that the map $\begin{cases} x = \lambda x' \\ y = \lambda y' \end{cases}$, *($\lambda > 0$) is an isometry of the Lobachevsky plane on itself*

b) Give an example of another isometry of Lobachevsky plane

The map $\begin{cases} x = \lambda x' \\ y = \lambda y' \end{cases}$, $\lambda > 0$ is diffeomorprhism, since this map is bijection, and the map and its iverse a map $\begin{cases} x' = \frac{x}{\lambda} \\ y' = \frac{y}{\mathrm{L}} \end{cases}$, $\lambda > 0$ are defined by smooth functions.

Show that diffeomorphism F this is iisometry

$$F^*(G) = F^*\left(\frac{dx^2 + dy^2}{x^2 + y^2}\right) = \frac{\left(d(\lambda x')\right)^2 + \left(d(\lambda y')\right)^2}{\left(\lambda x'\right)^2 + \left(\lambda y'\right)^2} = \frac{dx'^2 + dy'^2}{x'^2 + y'^2}.$$

One can easy to see that diffeomorphism $\begin{cases} x = x' + a \\ y = y' \end{cases}$ is also isometry. Little bit harder but still easy to check that a map

$$\begin{cases} x = \frac{x'}{x'^2 + y'^2} \\ y = \frac{y'}{x'^2 + y'^2} \end{cases}$$

is also diffeomorphism and this diffeomorphism is an isometry.

6 *Show that surface of the cone* $\begin{cases} x^2+y^2-k^2z^2=0 \\ z>0 \end{cases}$ *in* $\mathbf{E}^3$ *is locally Euclidean Riemannian surface, (is locally isometric to Euclidean plane).*

Find a relation of local Eucldean coordinates (u,v) *on the cone with coordinates* t,r *considered in question 3)*

This means that we have to find local coordinates u,v on the cone such that in these coordinates induced metric $G|_c$ on cone would have the appearance $G|_c = du^2+dv^2$.

First of all calculate the metric on cone in coordinates h,φ where

$$\mathbf{r}(h,\varphi)\colon \begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases}.$$

$(x^2+y^2-k^2z^2 = k^2h^2\cos^2\varphi + k^2h^2\sin^2\varphi - k^2h^2 = k^2h^2-k^2h^2 = 0.$

Calculate metric G_c on the cone in coordinates h,φ induced with the Euclidean metric $G = dx^2+dy^2+dz^2$:

$$G_{\text{cone}} = \left(dx^2+dy^2+dz^2\right)\big|_{x=kh\cos\varphi,y=kh\sin\varphi,z=h} = (k\cos\varphi dh - kh\sin\varphi d\varphi)^2 +$$

$$(k\sin\varphi dh + kh\cos\varphi d\varphi)^2 + dh^2 = (k^2+1)dh^2 + k^2h^2d\varphi^2\,.$$

Bearing in mind an analogy with polar coordinates try to find new local coordinates u,v such that

$$\begin{cases} u = \alpha h\cos\beta\varphi \\ v = \alpha h\sin\beta\varphi \end{cases},$$

where α and β are parameters. We come to

$$du^2+dv^2 = (\alpha\cos\beta\varphi dh - \alpha\beta h\sin\beta\varphi d\varphi)^2 + (\alpha\sin\beta\varphi dh + \alpha\beta h\cos\beta\varphi d\varphi)^2 = \alpha^2dh^2+\alpha^2\beta^2h^2d\varphi^2.$$

Comparing with the metric on the cone $G_{cone} = (1+k^2)dh^2+k^2h^2d\varphi^2$ we see that if we put $\alpha = \sqrt{k^2+1}$ and $\beta = \frac{k}{\sqrt{1+k^2}}$ then $du^2+dv^2 = \alpha^2dh^2+\alpha^2\beta^2h^2d\varphi^2 = (1+k^2)dh^2+k^2h^2d\varphi^2$.

Thus in new local coordinates

$$\begin{cases} u = \sqrt{k^2+1}h\cos\frac{k}{\sqrt{k^2+1}}\varphi \\ v = \sqrt{k^2+1}h\sin\frac{k}{\sqrt{k^2+1}}\varphi \end{cases}$$

induced metric on the cone becomes $G|_c = du^2+dv^2$, i.e. surface of the cone is locally isometric to the Euclidean plane (is locally Euclidean Riemannian surface).

Locally Euclidean coordinates u,v are related with conformal coordinates t,r by relation

$$u+iv = e^{t+ir}\,,\quad \text{i.e. } u = e^t\cos r\,, v = e^t\sin r\,.$$

$\blacksquare$