Solutions 7

1 *Show that vertical lines $x = a$ are geodesics (un-parameterised) on the Lobachevsky plane [1].*

* *Show that upper arcs of semicircles $(x-a)^2+y^2 = R^2, y > 0$ are (non-parametersied) geodesics.*

a) Consider second order differential equations defining geodesics with initial conditions such that "horisontal" velocity equals to zero: (we uses the information from Homework 6 or from Lecture notes about Christoffels for Lobachevsky plane: $\Gamma^x_{xx} = 0, \Gamma^x_{xy} = \Gamma^x_{yx} = -\frac{1}{y}$, $\Gamma^x_{yy} = 0$, $\Gamma^y_{xx} = \frac{1}{y}, \Gamma^y_{xy} = \Gamma^y_{yx} = 0, \Gamma^y_{yy} = -\frac{1}{y}$.)

$$\begin{cases} \ddot{x} - \frac{2\dot{x}\dot{y}}{y} = 0 \\ \ddot{y} + \frac{\dot{x}^2}{y} - \frac{\dot{y}^2}{y} = 0 \\ x(t)\big|_{t=t_0} = x_0, \dot{x}(t)\big|_{t=t_0} = 0 \\ y(t)\big|_{t=t_0} = y_0, \dot{y}(t)\big|_{t=t_0} = \dot{y}_0 \end{cases}$$

This equation has a solution and it is unique. One can see that if we put $x(t) \equiv 0$, i.e. curve is vertical then we come to the equation $\ddot{y} - \frac{\dot{y}^2}{y} = 0$. Solution of these equation gives curve $x = x_0$, $y = y(t)$: $\ddot{y} - \frac{\dot{y}^2}{y} = 0$. The image of this curve clearly is vertical ray $x = x_0, y > 0$.

b) $*$) the transformation $\begin{cases} x = \frac{u}{u^2+v^2} \\ y = \frac{v}{u^2+v^2} \end{cases}$ (inversion) is an isometry of Lobachevsky plane:

$$\frac{dx^2+dy^2}{y^2} = \frac{\left(d\left(\frac{u}{u^2+v^2}\right)\right)^2 + \left(d\left(\frac{v}{u^2+v^2}\right)\right)^2}{\left(\frac{v}{u^2+v^2}\right)^2} = \frac{\left[\frac{(v^2-u^2)du-2uvdv}{(u^2+v^2)^2}\right]^2 + \left[\frac{(u^2-v^2)du-2uvdv}{(u^2+v^2)^2}\right]^2}{\left(\frac{v}{u^2+v^2}\right)^2}$$

$$\frac{\frac{(v^2-u^2)^2du^2+4u^2v^2dv^2+(u^2-v^2)^2dv^2+4u^2v^2du^2}{(u^2+v^2)^4}}{\left(\frac{v}{u^2+v^2}\right)^2} = \frac{\frac{(v^2+u^2)^2du^2+(u^2+v^2)^2dv^2}{(u^2+v^2)^4}}{\left(\frac{v}{u^2+v^2}\right)^2} = \frac{du^2+dv^2}{v^2}\,.$$

Under the inversion the vertical line $\begin{cases} u = 1 \\ v = t \end{cases}$, geodesic, transforms to the half-circle with the centre at the point $\left(\frac{1}{2}.0\right)$ of absolute. Hence this half-circle is geodesic. Using a suitable translation $x \mapsto x + a$ and dilation one can obtain arbitrary half-circle with centre at the absolute from this half-circle.

---

[1] *As usual we consider here a realisation of the Lobachevsky plane (hyperbolic plane) as upper half of Euclidean plane $\{(x, y): y > 0\}$ with the metric $G = \frac{dx^2+dy^2}{y^2}$. The line $y = 0$ is called absolute.*

2 *Consider a vertical ray* $C: x(t) = 1, y(t) = 1 + t$, $0 \le t < \infty$ *on the Lobachevsky plane.*

Find the parallel transport $\mathbf{X}(t)$ *of the vector* $\mathbf{X}_0 = \partial_y$ *attached at the initial point* $(1,1)$ *along the ray* C *at an arbitrary point of the ray.*

Find the parallel transport $\mathbf{Y}(t)$ *of the vector* $\mathbf{Y}_0 = \partial_x + \partial_y$ *attached at the same initial point* $(1,1)$ *along the ray* C *at an arbitrary point of the ray. (Exam question, 2013.)*

Since vertical ray is geodesic then during parallel transport vector $\mathbf{X}(t)$ remains proportional to velocity vector. Hence $\mathbf{X}(t) = k(t)\partial_y$. On the other hand during parallel transport its length is not changed, since the connection is Levi-Civita connection. i.e. scalar product

$$\langle \mathbf{X}(t), \mathbf{X}(t) \rangle = \langle k(t)\partial_y, k(t)\partial_y \rangle = \frac{k^2(t)}{(y_0 + t)^2} = \frac{k^2(t)}{(1+t)^2} = Constant$$

At the moment $t = 0$ it is equal to $\frac{1}{1} = 1$. We have $\frac{k^2(t)}{(1+t)^2} = 1$, i.e. $k(t) = \pm(1+t)$

Since at $t = 0$, $k = 1$ we choose sign $+$ and $k(t) = 1 + t$. We come to $\mathbf{X}(t) = (1+t)\partial_y$

During parallel transport of two vectors along Levi-Civita connection not only their lengths but angles between them is not changed too.

Consider vector $\mathbf{Z} = \partial_x$. it is orthogonal to vector $\mathbf{X}$. Hence during parallel transport it will remain orthogonal. Hence $\mathbf{Z}(t) = k'(t)\partial_x$ since vectors ∂_x, ∂_y are orthogonal to each other at any point of the Lobachevsky plane. The length fo the vector $\mathbf{Z}(t)$ is preserved too. Hence it has to be equal always to 1 since at $t = 0$ it is equal to 1. We come to $\mathbf{Z}(t) = (1+t)\partial_x$.

Now by linearity of parallel transport $\mathbf{Y}(t) = \mathbf{X}(t) + \mathbf{Z}(t) = (1+t)(\partial_x + \partial_y)$.

3 *Find a parameterisation of vertical lines in the Lobachevsky plane such that they become parameterised geodesics.*

We know also that vertical line is geodesic. Let $x = x_0, y = f(t)$ be right parameterisation, i.e. parameterisation such that veclotiy vector remains velocity vector during parallel transport. Velocity vector $\mathbf{v}(t) = \begin{pmatrix} 0 \\ f_t \end{pmatrix}$. Its length is equal to $\sqrt{\frac{x_t^2 + y_t^2}{y^2}} = \sqrt{\frac{0 + f_t^2}{f^2}} = \frac{f_t}{f}$ and it has remain the same. Hence $\frac{f_t}{f} = c$, i.e. $f(t) = Ae^{ct}$. We see that $x = x_0, y = ae^{ct}$ is parameterised geodesic. (One can see that differential equation of geodesics are obeyed (see the exercise 1)).

4 *Find geodesics on cylinder*

a) using straightforwardly equations of geodesics

b) using the properties of acceleration vector for particle movin along geodesic

c) using the fact that geodesic is shortest

In all the cases state clearly, Is it parameterised geodesic, or un-parameterised.

a,b) see the lecture notes

c) Developping the surface of the cylinder we come to the plane. The lines on the planes are helices on the cylindre. These are geodesics.

5 *Great circle is a geodesic.*

Every geodesic is a great circle.

What curves are these statements about, parameterised or un-parametersied?

We have to distinguish between parameterised and un-parameterised geodesics. The precise statments are: great circles indeed are *un-parametersied* geodesics. One can consider suitable parameterisation of great circle such that it becomes geodesic (i.e. parameterised geodesics) For this purpose one has to consider a parameterisation such that speed is constant in this parameterisation.

Every geodesic on sphere conisdered as unparametersied curve is great circle.

(See in detail lecture notes.)

6 *Let (M, G) be a Riemannian manifold. Let C be a curve on M starting at the point* $\mathbf{pt}_1$ *and ending at the point* $\mathbf{pt}_2$*.*

Define an operator $P_C: T_{\mathbf{pt}_1}M \to T_{\mathbf{pt}_2}M$.

Explain why the parallel transport P_C is a linear orthogonal operator.

Let the points $\mathbf{pt}_1$ and $\mathbf{pt}_2$ coincide, so that C is a closed curve.

Let $\mathbf{a}$ be a vector attached at the point $\mathbf{pt}_1$, and $\mathbf{b} = P_C(\mathbf{a})$.

Consider operator P_C^2. Suppose that $P_C(\mathbf{a}) = \mathbf{b}$ and $P_C^2(\mathbf{a}) = -\mathbf{a}$. Show that vectors $\mathbf{a}$ *and* $\mathbf{b}$ *are orthogonal to each other. (Exam question 2016)*

Since parallel transport is orthogonal operators, hence it preserves the scalar product, i.e. $< \mathbf{a}, \mathbf{b} >=< P_C\mathbf{a}, P_C\mathbf{b} >=< \mathbf{b}, -\mathbf{a} >= - < \mathbf{a}, \mathbf{b} >$, i.e. $< \mathbf{a}, \mathbf{b} >= 0$, i.e. these vectors are orthogonal to each other.

7 *On the unit sphere $x^2 + y^2 + z^2 = 1$ in $\mathbf{E}^3$ consider the curve C defined by the equation $\cos\theta - \sin\theta\sin\varphi = 0$ in spherical coordinates.*

Show that in the process of parallel transport along the curve C an arbitrary tangent vector to the curve remains tangent to the curve. (Exam question 2016)

Notice that $\cos\theta - \sin\theta\sin\varphi = x - z\big|_{x^2+y^2+z^2=1}$, i.e. the curve C is the intersection of the plane $x - z$ which goes through origin with the sphere. This means that C is great circle. Hence tangent vector remains tangent (and keeping its length) due to parallel transport.

8 *On the sphere $x^2 + y^2 + z^2 = R^2$ of radius R in $\mathbf{E}^3$ consider the following three closed curves.*

a) the triangle $\triangle ABC$ with vertices at the points $A = (0, 0, 1)$, $B = (0, 1, 0)$ and $C = (1, 0, 0)$. The edges of triangle are geodesics.

b) the triangle $\triangle ABC$ with vertices at the points $A = (0, 0, 1)$, $B = (0, \cos\varphi, \sin\varphi)$ and $C = (1, 0, 0)$, $0 < \varphi < \frac{\pi}{2}$ The edges of triangle are geodesics.

c) the curve $\theta = \theta_0$ (line of constant latitude).

Consider the result of parallel transport of the vectors tangent to sphere over these closed curves.

In the case a) angle of rotation will be $\frac{\pi}{2}$: e,g, take the vector ∂_y at the point A and see how it transforms during parallel transport. This vector is tangent to the sphere. (The angle of rotation is the same for all the tangent vectors.)

The vector ∂_y during parallel transport along the arc AB will remain tangent to this arc, since it is an arc of geodesic, and it will preserve its length. Hence at the point B it will become the vector ∂_z.

The vector ∂_z during parallel transport along the arc of geodesic BC will remain always orthogonal to this arc, since at the initial point it was orthogonal to the arc, i.e. it will remian the same vector (in the ambient space $\mathbf{E}^3$), and at the point C it will remain ∂_z,

Then the same reasoning for the curve CA: during parallel transport along arc of geodesic CA it will remain always tangent to the curve, and finally it will be the vector ∂_x. We see that the vector ∂_y becomes the vector ∂_x after parallel transport.

Notice that angle of rotation

$$\frac{\pi}{2} = \frac{\text{area of the triangle } ABC}{R^2} = K \cdot \text{area of the triangle } ABC\,.$$

For the case b) doing the same considerations we come to the fact that vector rotates on the angle φ.

One can take initial vector $\mathbf{A} = \cos\varphi\partial_x + \sin\varphi\partial_y$. During parapllel transport this vector will remain tangent to the arc AB since it is geodesic, and its length will not change. The result of parallel trasport at the point B will be the vector ∂_z. The vector ∂_z during parallel transport along the arc of geodesic BC will remain always orthogonal to this arc, and in the same way as in the case a) we will come to the conclusion that at the point C it will remain ∂_z. Then applying the same reasoning as in the case a) we will come to the conclusion that finally it will be the vector ∂_x. We see that the vector ∂_y becomes the vector ∂_x after parallel transport.

Notice that angle of rotation

$$\frac{\varphi}{2} = \frac{\text{area of the triangle } ABC}{R^2} = K \cdot \text{area of the triangle } ABC\,.$$

In the case c) the closed curve C is not geodesic, and we have to apply the Theorem The curve C is the boundary of the segment of the sphere D:

$$D = \{x, y, z \colon R\cos\theta_0 \leq z \leq R\}$$

The height of this domain is equal to $h = R(1 - \cos\theta_0)$, and The area of this domain is equal to $S = 2\pi Rh = 2\pi R^2(1 - \cos\theta_0)$. Due to the Theorem of parallel transport we have for angle of rotation $\Delta\Phi$

$$\Delta\Phi = \int_{R\cos\theta_0 \leq z \leq R} K d\sigma = K\cdot\text{Area of the segment} = 2\pi R^2(1 - \cos\theta_0)\mathbf{R}^2 = 2\pi(1-\cos\theta_0)\,. \blacksquare$$