Solutions 9

1 *On the sphere $x^2+y^2+z^2=R^2$ in $\mathbf{E}^3$ consider a circle C which is the intersection of the sphere with the plane $z=R-h$, $0<h<R$*

Let $\mathbf{X}$ be an arbitrary vector tangent to the sphere at a point of C.

Find the angle between $\mathbf{X}$ and the result of parallel transport of $\mathbf{X}$ along C.

The circle C is a boundary of the sphere segment of the height H. The area of this domain is equal to $2\pi Rh$. The Gaussian curvature of sphere iis equal to $K=\frac{1}{R^2}$. Hence due to Theorem we see that vector $\mathbf{X}$ through parallel transport rotates on the angle $KS=\frac{2\pi R}{h}$.

2 *Write down components of the curvature tensor R^i_{kmn} in terms of the Christoffel symbols Γ^i_{km} and its derivatives.*

$$R^i_{kmn}\partial_i = \mathcal{R}(\partial_m,\partial_n)\partial_k = \nabla_m\left(\nabla_n\partial_k\right) - (m\leftrightarrow n) = \nabla_m(\Gamma^r_{nk}\partial_r) - (m\leftrightarrow n) =$$

$$\left(\partial_m\Gamma^i_{nk} + \Gamma^i_{mr}\Gamma^r_{nk}\right)\partial_i - (m\leftrightarrow n)$$

i.e.

$$R^i_{kmn} = \partial_m\Gamma^i_{nk} + \Gamma^i_{mr}\Gamma^r_{nk} - \partial_n\Gamma^i_{mk} - \Gamma^i_{nr}\Gamma^r_{mk}\,.$$

(See also lecture notes).

3 *For every of the statements below prove it or show that it is wrong considering counterexample.*

a) If there exist coordinates u,v such that Riemannian metric G at the given point $\mathbf{p}$ is equal to $G=du^2+dv^2$ in these coordinates, then the Riemann curvature tensor at the point $\mathbf{p}$ vanishes.

b) If all first derivatives of components of Riemannian metric in coordinates u,v vanish at the given point $\mathbf{p}$:

$$\frac{\partial g_{ik}(u,v)}{\partial u}\Big|_{\mathbf{p}} = \frac{\partial g_{ik}(u,v)}{\partial v}\Big|_{\mathbf{p}} = 0\,,$$

then the Riemann curvature tensor also vanishes at this point.

c) If all first and second derivatives of components of Riemannian metric

$$\frac{\partial g_{ik}(u,v)}{\partial u}\Big|_{\mathbf{p}} = \frac{\partial g_{ik}(u,v)}{\partial v}\Big|_{\mathbf{p}} = \frac{\partial^2 g_{ik}(u,v)}{\partial u^2}\Big|_{\mathbf{p}} = \frac{\partial^2 g_{ik}(u,v)}{\partial u\partial v}\Big|_{\mathbf{p}} = \frac{\partial^2 g_{ik}(u,v)}{\partial v^2}\Big|_{\mathbf{p}} = 0$$

vanish at the given point then the Riemann curvature tensor also vanishes at this point.

First and second statements are wrong. The thrid statement is true.

Counterexample to the first statement: Consider on the unit sphere metric $G = d\theta^2 + \sin^2\theta d\varphi^2$.

At the points of equator (but not in their neighborhood!!!!) this metric is Euclidean and first derivatives of components vanish, but curvature is not vanished.

Counterexample to the second statement:

again we can consider the points of equator derivatives of metric's components vanish, at points of equator, but curvature is not vanished (see in detail the next exercise)

One can consider another more general example:

Consider surface $\mathbf{r} = \mathbf{r}(u, v)$: $x = u, y = v, z = F(x, y)$. Induced Riemannian metric $G = g_{\alpha\beta}du^\alpha du^\beta = g_{uu}du^2 + 2g_{uv}dudv + g_{vv}dv^2$

$$G = \begin{pmatrix} 1 + F_u^2 & F_u F_v \\ F_u F_v & 1 + F_v^2 \end{pmatrix}$$

One can see that at the points of extremum of function F first derivatives of Riemannian metric vanish, this Christoffel symbols vanish at the extrema in coordinates u, v, but the curvature in general does not vanish (It is proportional to $F_{uu}F_{vv} - F_{uv}^2$.)

If at the given point first and second derivatves of metric vanish then due to Levi-Civita formula Christoffel symbols and their first derivatives vanish. This imply that curvature vanish too.

4 *Let $x^i, i = 1, \dots, n$ be local coordinates on Riemannian manifold M such that for Riemannian metric tensor $G = g_{ik}(x)dx^i dx^k$ the following condition holds: first derivatives of all components of metric tensor vanish at the given point $\mathbf{p}$:*

$$\frac{\partial g_{ik}(x)}{\partial x^m}\Big|_{\mathbf{p}} = 0 \ \ (i, k, m = 1, \dots, n)\,. \tag{4.1}$$

Write down components R^i_{kmn} of the Riemann curvture tensor in terms of Christoffel symbols Γ^i_{km} and its derivatives at the point $\mathbf{p}$ in these local coordinates

Find points on the sphere of radius a in $\mathbf{E}^3$ such that condition (4) holds in spherical coordinates, and calculate Riemann curvature tensor in these points of sphere

Calculate Riemann scalar curvature at arbitrary point of the sphere.

Compare results of calculations with formula for relation between the Gaussian curvature and Riemann curvature tensor for surfaces in $\mathbf{E}^3$:

$$K = \frac{R}{2} = \frac{R_{1212}}{\det g}\,. \tag{4.1a}$$

a) this condition implies that the Christoffel symbols vanish at the point $\mathbf{p}$, hence

$$R^i_{kmn} = \partial_m \Gamma^i_{nk} - \partial_n \Gamma^i_{mk}\,. \tag{4.2}$$

b) in spherical coordinates (θ,φ) $G = a^2(d\theta^2 + \sin^2\theta d\varphi^2)$,

$$g_{ik} = \begin{pmatrix} a^2 & 0 \\ 0 & a^2\sin^2\theta \end{pmatrix}$$

we see that $\frac{\partial(a^2\sin^2\theta}{\partial\theta)} = 2a^2\sin\theta\cos\theta$, and all other derivatives vanish, hence at points where $\cos\theta = 0$, i.e at points of equator Christoffel symbols vanish. Using formula (4.1) and Levi-Civita formula for Christoffel symbols we see that

$$\Gamma^\theta_{\varphi\varphi} = \frac{1}{2}g^{\theta\theta}\left(-\frac{\partial g_{\varphi\varphi}}{\partial\theta}\right) = \frac{1}{2a^2}\left(-2a^2\sin\theta\cos\theta\right) = -\sin\theta\cos\theta\,,$$

and

$$\Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = \frac{1}{2}g^{\varphi\varphi}\left(\frac{\partial g_{\theta\varphi}}{\partial\theta}\right) = \frac{1}{2a^2\sin^2\theta}\left(2a^2\sin\theta\cos\theta\right) = \cot\theta,,$$

all other and for components of Riemann tensor at the points where $\theta = \frac{\pi}{2}$ (points of equator) we have:

$$R^\theta{}_{\varphi\theta\varphi}\big|_{\theta=\pi/2} = \left(\partial_\theta\Gamma^\theta_{\varphi\varphi} - \partial_\varphi\Gamma^\theta_{\theta\varphi}\right)\big|_{\theta=\pi/2} = 1\,, \Rightarrow R_{\theta\varphi\theta\varphi}\big|_{\theta=\pi/2} = \frac{1}{a^2}\,.$$

$$R^\varphi{}_{\theta\varphi\theta}\big|_{\theta=\pi/2} = \left(\partial_\varphi\Gamma^\varphi_{\theta\theta} - \partial_\theta\Gamma^\varphi_{\varphi\theta}\right)\big|_{\theta=\pi/2} = 1\,, \Rightarrow R_{\theta\varphi\theta\varphi}\big|_{\theta=\pi/2} = \frac{1}{a^2}\,.$$

We see that Riemann scalar curvature at the points of equator

$$R\big|_{\theta=\pi/2} = R^i{}_{min}g^{mn}\big|_{\theta=\pi/2} = \left(R^\theta_{\varphi\theta\varphi}g^{\varphi\varphi} + R^\varphi_{\theta\varphi\theta}g^{\theta\theta}\right)\big|_{\theta=\pi/2} = \frac{2}{a^2}\,.$$

(we omit all zero components)

Let $\mathbf{p}$ be an arbtrary point. We can choose coordinates such that at this point $\theta = \frac{\pi}{2}$, hence the scalar curvature at any point is equal to $\frac{2}{a^2}$.

We know that the Gaussian curvature the Riemann curvature tensor for Levi-Civita connection:

$$K = \frac{R}{2} = \frac{R_{1212}}{\det g}\,.$$

Scalar curvature is equal at all the points of the sphere to $\frac{2}{a^2}$, the Gaussian curvature is equal to $\frac{1}{a^2}$, and $\det g = a^4\sin^2\theta$, thus we see that $R_{\theta\varphi\theta\varphi} = a^2\sin^2\theta$.