COURSEWORK (Short Solutions)
1

(a) *Define atlases on real projective line $\mathbb{R}P^1$ and the circle S^1.*

(b) *Establish a diffeomorphism between $\mathbb{R}P^1$ and S^1.*

(c) *Consider the vector $\mathbf{e}_\theta = \frac{\partial}{\partial\theta}$ tangent to the circle at the point θ_0 (θ is a polar coordinate). Write down the explicit expression in local coordinates on $\mathbb{R}P^1$ of the image of this vector under the above diffeomorphism.*

a) Consider $\mathbb{R}P^1$ as a set of lines passing through the origin of $\mathbb{R}^2$. Define the atlas which possesses two charts $(U_1, \varphi_1), ((U_2, \varphi_2))$, where U_1 is the set of all the lines which intersect the line $y = 1$, U_2 is the set of all the lines which intersect the line $x = 1$. The value of φ_1 on any element in U_1 is equal to the x-coordinate of intersection of the line with the line $y = 1$. Respectively the value of φ_2 on any element in U_2 is equal to the y-coordinate of intersection of the line with the line $x = 1$.

To define the atlas on the circle S^1 we consider two charts— stereographic coordinates on the circle $x^2 + y^2 = 1$ related with North Pole $u = \frac{x}{1-y}$ and with South Pole $u' = \frac{x}{1+y}$ (see in detail Homework 2).

b) All points $(x, y) \in S^1$ except north pole are in one-one correspondence with $\mathbb{R}^1$ by using stereographic projection $u = \frac{x}{1-y}$. The points of $\mathbb{R}^1$ are in one-one correspondence with the points of $\mathbb{R}P^1$ which belong to the chart U_1, i.e. with the lines which intersect the line $y = 1$. We come to the map $F(x, y) = [1 : \frac{x}{1-y}] = [1 - y : x]$ from the chart $S^1 \backslash N$ onto the chart U_1 on $\mathbb{R}P^1$

All points $(x, y) \in S^1$ except south pole are in one-one correspondence with $\mathbb{R}^1$ by using stereographic projection $u' = \frac{x}{1+y}$. The points of $\mathbb{R}^1$ are in one-one correspondence with the points of $\mathbb{R}P^1$ which belong to the chart U_2, i.e. with the lines which intersect the line $x = 1$. We come to the map $F(x, y) = [\frac{x}{1+y} : 1] = [x : 1 + y]$ from the chart $S^1 \backslash S$ onto the chart U_2 on $\mathbb{R}P^1$

The diffeomorphism $F(x, y)$ for an arbitrary point $(x, y) \in S^1$ can be defined by the formulae

$$F(x, y) = \begin{cases} [1 - y : x] \text{ if } y \neq 1 \\ [x : 1 + y] \text{ if } y \neq -1 \end{cases}$$

c) $\mathbf{e}_\theta = \frac{\partial}{\partial\theta}$. We have that $u = \frac{x}{1-y} = \frac{\sin\theta}{1-\cos\theta}$. Hence $u(\theta) = \frac{\sin\theta}{1-\cos\theta}$ and

$$\mathbf{e}_\theta = \partial_\theta = \frac{\partial u(\theta)}{\partial\theta}\partial_u = -\frac{1}{1 - \cos\theta}\partial_u$$

2

Consider the real projective plane $\mathbb{R}P^2$.

(a) *Consider a natural atlas for $\mathbb{R}P^2$ consisting of three charts. Write the charts explicitly, showing their domains and codomains.*

(b) *Calculate the changes of coordinates between all charts of this atlas.*

(c) *Introduce the coordinate basis of tangent vectors associated with each chart and calculate the transformation between such bases for each pair of charts.*

We consider $\mathbb{R}P^2$ as a set of equivalence classes of $[x : y : z]$ of non-zero vectors in $\mathbb{R}^3$

Consider the charts $(U_1, \varphi_1), (U_2, \varphi_2), (U_3, \varphi_3)$ such that

$$U_1 = \{[x : y : z] : x \neq 0 \;\; \varphi_1 \left([x : y : z]\right) \mapsto (u_1, v_1)\}, \qquad \begin{cases} u_1 = \frac{y}{x} \\ v_1 = \frac{z}{x} \end{cases}$$

$$U_2 = \{[x : y : z] : y \neq 0 \;\; \varphi_2 \left([x : y : z]\right) \mapsto (u_2, v_2)\}, \qquad \begin{cases} u_2 = \frac{x}{y} \\ v_2 = \frac{z}{y} \end{cases}$$

$$U_3 = \{[x : y : z] : z \neq 0 \;\; \varphi_3 \left([x : y : z]\right) \mapsto (u_3, v_3)\}, \qquad \begin{cases} u_3 = \frac{x}{z} \\ v_3 = \frac{y}{z} \end{cases}$$

Now calculate the transformations of local coordinates (u_1, v_1) to (u_2, v_2) and corresponding tangent vectors $\frac{\partial}{\partial u_1}, \frac{\partial}{\partial v_1}$ to $\frac{\partial}{\partial u_2}, \frac{\partial}{\partial v_2}$.

Note that

$$u_1 = \frac{1}{u_2}, \;\; v_1 = \frac{v_2}{u_2}$$

For basis vectors using chain rule we have:

$$\frac{\partial}{\partial u_2} = \frac{\partial u_1}{\partial u_2}\frac{\partial}{\partial u_1} + \frac{\partial v_1}{\partial u_2}\frac{\partial}{\partial v_1}$$

and

$$\frac{\partial}{\partial v_2} = \frac{\partial u_1}{\partial v_2}\frac{\partial}{\partial u_1} + \frac{\partial v_1}{\partial v_2}\frac{\partial}{\partial v_1} \, .$$

3

Consider a set M in $\mathbb{R}^4$ given by the equations

$$\begin{cases} x^2 + y^2 + z^2 + t^2 = 1 \\ x + y + z + t = 0 \end{cases}$$

(a) *Show that this set is a smooth manifold.*

Consider the vectors

$$\mathbf{A} = \partial_x - \partial_y - \partial_z + \partial_t,\ \mathbf{B} = b\partial_x - a\partial_y + a\partial_z - b\partial_t,\ \mathbf{C} = \partial_x + \partial_y + \partial_z + \partial_t,$$

at the point $\mathbf{x}_0 = (a, b, -b, -a)$*, where* a, b *are parameters such that* $(2a^2 + 2b^2 = 1)$*.*

(b) *Show that the vectors* $\mathbf{A}$ *and* $\mathbf{B}$ *are tangent to this manifold and vector* $\mathbf{C}$ *is not tangent to this manifold.*

(c) *Find all the vectors which are tangent to this manifold at the point* $\left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$*.*

Consider the matrix of derivatives

$$\begin{pmatrix} 2x & 2y & 2z & 2t \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

on the points where these equations are equal simultaneously to zero.

The rank of this matrix is greater or equal to 1 because the second vector is not zero. It is equal to 1 at the points where $x = y = z = t = c \neq 0$. But at these points $x+y+z+t \neq 0$. Hence we see that rank of the matrix is equal to 2 at zeros of equation. Hence system of equations defines manifold of dimension $4 - 2 = 2$. (It is sphere.)

To see that vectors $\mathbf{A}, \mathbf{B}$ are tangent to this manifold and the vector $\mathbf{C}$ is not tangent we have to act by corresponding differential operator:

$$\mathbf{A}(x^2 + y^2 + z^2 + t^2 - 1)|_{\mathbf{x}_0} = (2x - 2y - 2z + 2t)|_{\mathbf{x}_0} = (2a + 2b - 2b - 2a) = 0,$$

and $\mathbf{A}(x + y + z + t)|_{\mathbf{x}_0} = 1 - 1 - 1 + 1 = 0$. For $\mathbf{B}$:

$$\mathbf{B}(x^2 + y^2 + z^2 + t^2 - 1)|_{\mathbf{x}_0} = 2ab - 2ba - 2ab + 2ba = 0, \qquad \mathbf{B}(x + y + z + t)|_{\mathbf{x}_0} = 0$$

$\mathbf{C}(x + y + z + t) = 4$ for all the points. Hence it is not tangent.

c)

The vector field $\mathbf{X} = a\partial_x + b\partial_y + c\partial_z + d\partial_t$ which is tangent to the manifold at the point $\{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}$ obey the following conditions:

$$\begin{cases} a - b - c + d = 0 \\ a + b + c + d = 0 \end{cases}, \quad \text{, i.e.} \qquad \mathbf{X} = a(\partial_x - \partial_t) + b(\partial_y - \partial_z)$$

4

(a) *Show that* $SL(2)$ *is a manifold. (*$SL(2)$ *is a group of* 2×2 *matrices with determinant equal to* 1*.)*

(b) *Describe the tangent space $T_E SL(2)$ to this manifold at the identity. In particular find a basis in the tangent space. (You should represent tangent vectors as matrices.)*

(c) *For the matrix $A = \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$ show that* $\exp A \in SL(2)$ *and calculate* $\exp A$.

(Recall that for an arbitrary matrix X, $\exp X = \sum_{n=0}^{\infty} \frac{X^n}{n!} = 1 + X + \frac{X^2}{2} + \frac{X^3}{6} + \frac{X^4}{24} + \dots$*)*

$SL(2)$ is defined in $\mathbb{R}^4$ by equation $F(x, y, z, t) = \det \begin{pmatrix} x & y \\ z & t \end{pmatrix} = xt - zy = 1$. Consider the vector of derivatives

$$\left(\frac{\partial F(x, y, z, t)}{\partial x}, \frac{\partial F(x, y, z, t)}{\partial x}, \frac{\partial F(x, y, z, t),}{\partial x}, \frac{\partial F(x, y, z, t)}{\partial x} \right) = (t, -z, -y, x)$$

One can see that this vector does not vanish on the zeros of the function F. (formally speaking matrix has the rank 1). Indeed if $xt - yz = 1$ then if $x = 0$ then $z \neq 0$ in a vicinity of a given point. Hence we come to smooth manifold of dimension $4 - 1 = 3$.

b) Let $\mathbf{A} = a\partial_x + b\partial_y + c\partial_z + d\partial_t$ be tangent vector at the point (x_0, y_0, z_0, t_0) of the manifold (group). Then

$$\mathbf{A}F = \mathbf{A}(xt - yz - 1) = (a\partial_x + b\partial_y + c\partial_z + d\partial_t)(xt - yz) = at_0 - bz_0 - cy_0 + dx_0 = 0\,.$$

We come to the statement: The vector $\mathbf{A} = a\partial_x + b\partial_y + c\partial_z + d\partial_t$ is tangent to $SL(2)$ at the point $(x_0, y_0, z_0, t_0) \in SL(2)$ if $at_0 - bz_0 - cy_0 + dx_0 = 0$. In particular if we consider identity point of $SL(2)$ when $A = \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $(x_0 = t_0 = 1, y_0 = z_0 = 0)$, then we come to the condition: $a + d = 0$ i.e. trace matrices in Lie algebra is equal to zero.

(c)

First solution.

Consider matrix $A = \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$. We see that $A^2 = a^2 I$. Hence $A^{2n} = a^{2n} I$ and $A^{2n+1} = a^{2n} \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$. We have

$$B = e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I \sum_{n=0}^{\infty} \frac{a^{2n}}{(2n)!} + \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} \sum_{n=0}^{\infty} \frac{a^{2n}}{(2n+1)!} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

where

$$B_{11} = \sum \frac{a^n}{n!} = e^a, \quad B_{22} = \sum \frac{(-1)^n a^n}{n!} = e^{-a}, \quad B_{21} = 0,$$

$$B_{12} = b \left(1 + \frac{a^2}{3!} + \frac{a^4}{5!} + \frac{a^6}{7!} + \dots \right) = \frac{b}{a} \left(a + \frac{a^3}{3!} + \frac{a^5}{5!} + \frac{a^7}{7!} + \dots \right) = \frac{b}{a} \sinh a$$

Hence

$$e^{\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}} = \begin{pmatrix} e^a & \frac{b}{a}\sinh a \\ 0 & e^{-a} \end{pmatrix}$$

Another solution

Diagonalise matrix A. Find eigenvalues and eigenvectors of the matrix A. Consider matrix $X = \begin{pmatrix} 1 & \frac{b}{2a} \\ 0 & \frac{1}{2a} \end{pmatrix}$. (Columns of the matrix X are eigenvectors of the matrix A). We have that

$$A = \begin{pmatrix} 1 & -b \\ 0 & 2a \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{2a} \\ 0 & \frac{1}{2a} \end{pmatrix} = \begin{pmatrix} e^a & \frac{b}{a}\sinh a \\ 0 & e^{-a} \end{pmatrix}$$

Now everything is evident:

$$A^n = \begin{pmatrix} 1 & -b \\ 0 & 2a \end{pmatrix} \begin{pmatrix} a^n & 0 \\ 0 & (-a)^n \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{2a} \\ 0 & \frac{1}{2a} \end{pmatrix}$$

and

$$e^A = \begin{pmatrix} 1 & -b \\ 0 & 2a \end{pmatrix} \begin{pmatrix} e^a & 0 \\ 0 & e^{-a} \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{2a} \\ 0 & \frac{1}{2a} \end{pmatrix} = \begin{pmatrix} e^a & \frac{b(e^a - e^{-a})}{2a} \\ 0 & e^{-a} \end{pmatrix}$$

5

It is a known fact that the function $\Phi(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$ is a smooth function on $\mathbb{R}$, i.e. it has derivatives of all orders at an arbitrary point of $\mathbb{R}$.

(a) *Check directly that $\Phi(x)$ has at least the first and second derivatives at all points of $\mathbb{R}$ including the point $x = 0$.*

If $x > 0$ then obviously function $\Phi(x)$ is smooth. The first derivative of this function at zero by definition is equal to the $\lim_{x\to 0} \frac{\Phi(x)-\Phi(0)}{x}$ if it exists. One can check this directly using formulae of calculus. The same for second derivative.

(b) *Using this function construct explicitly a function $G(x)$ on $\mathbb{R}$ such that*

$$0 \le G(x) \le 1 \text{ and } G(x) = \begin{cases} 1 & \text{if } x \ge \frac{1}{2} \\ 0 & \text{if } x \le -\frac{1}{2} \end{cases}.$$

Consider the function

$$G(x) = \frac{\Phi\left(x + \frac{1}{2}\right)}{\Phi\left(x + \frac{1}{2}\right) + \Phi\left(\frac{1}{2} - x\right)}$$

One can see that this is the function which we want.