Homework 1

1 a) Using the stereographic projection from the north pole $N = (0,1)$ introduce stereographic coordinate for the part of the circle S^1 $(x^2+y^2=1)$ without the north pole.

b) Do the same but using the south pole $S=(0,-1)$ instead of the north pole.

c) express the stereographic coordinates obtained in a) and b) in terms of the angle φ (for polar coordinates in $\mathbf{R}^2$, $x = r\cos\varphi, y = r\sin\varphi$).

2 a) Using the stereographic projection from the north pole $N = (0,0,1)$ introduce stereographic coordinates (u,v) for the part of the sphere S^2 $(x^2+y^2+z^2=1)$ without the north pole.

b) Do the same but introduce stereographic coordinates (u',v') using the south pole $S=(0,0,-1)$ instead of the north pole.

c) express these stereographic coordinates via spherical coordinates θ,φ (for spherical coordinates in $\mathbf{R}^3$, $x = r\sin\theta\cos\varphi, y = r\sin\theta\sin\varphi,\ z = r\cos\theta$).

3* Introduce stereographic coordinates on n-dimensional sphere S^n.

4† Show that stereographic projection is the bijection between rational points on the unit sphere (the points on the unit sphere with rational coordinates) and rational points on the plane.

Find all rational solutions of the equations $x^2+y^2=1$, $x^2+y^2+z^2=1$.

Find all integer solutions of the equation $x^2+y^2=z^2$ (Pythagoreans triples.)

5 Considering the natural bijection of the part of real projective space [1)] $\mathbf{R}P^n$ on the plane $x^{n+1}=1$ in $\mathbf{R}^{n+1}$ ($(x^1,\dots,x^n,x^{n+1})$ are cartesian coordinates on $\mathbf{R}^{n+1}$) one can introduce "inhomogeneous coordinates" on the part of the $\mathbf{R}P^n$.

a) do this exercise for $n=1,2$; b)* do this exercise for an arbitrary n.

6 Find different "inhomogeneous" coordinates on $\mathbf{R}P^2$ considering natural bijections of the parts of $\mathbf{R}P^2$ on the plane $x^3=1$ or on the plane $x^2=1$ or on the plane $x^1=1$. Find relations between these coordinates.

7 Do the previous exercise for complex projective spaces $\mathbf{C}P^1$ and $\mathbf{C}P^2$.

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1) Real projective space $\mathbf{R}P^n = \{$set of straight lines in $\mathbf{R}^{n+1}$ which pass throug the origin$\}$, or in other words $\mathbf{R}P^n$ is the set of equivalence classes of non-zero vectors in $\mathbf{R}^{n+1}$, i.e. $\mathbf{R}P^n = \{[x^1 : x^2 : \dots : x^{n+1}\}]$ where $[x^1 : x^2 : \dots : x^{n+1}]$ stands for the equivalence class of a non-zero vector $(x^1,\dots,x^{n+1}) \in \mathbf{R}^{n+1}$.—Two non-zero vectors $(x^1,\dots,x^{n+1})$, $(x'^1,\dots,x'^{n+1})$ are considered equivalent if they are proportional: $x'^1 = \lambda x^1$, $x'^2 = \lambda x^2,\dots,x'^n = \lambda x^n, x'^{n+1} = \lambda x^{n+1}$.