Homework 2

1 Check whether the following subsets are open

a) the subset $\{(x,y)\in\mathbf{R}^2\colon\ x^2+y^2<1\}$,

b) the subset $\{(x,y)\in\mathbf{R}^2\colon\ x^2+y^2\leq 1\}$,

c) the subset $\{(x,y)\in\mathbf{R}^2\colon\ 0<x<1\}$

d) the subset $\{(x,y,z)\in\mathbf{R}^3\colon\ x^2+y^2<1, |z|<1\}$

e) the subset $\mathbf{R}^2\backslash I_-$, where $I_-=\{(x,y)\colon\ y=0, x\leq 0\}$.

2 Consider the sets U_1, U_2 on $\mathbf{R}^2$ such that $U_1=\mathbf{R}^2\backslash I_-$, $U_2=\mathbf{R}^2\backslash I_+$, where $I_-=\{(x,y)\colon\ y=0, x\leq 0\}$, $I_+=\{(x,y)\colon\ y=0, x\geq 0\}$. Using polar coordinates define charts $(U_1,\varphi_1),(U_2,\varphi_2)$. Show that these charts do not form an atlas on $\mathbf{R}^2$. Consider an additional chart (U_3,φ_3), where $U_3=\mathbf{R}^2$, $\varphi_3=\mathbf{id}$. Show that $\{(U_3,\varphi_3),(U_1,\varphi_1)\}$ is an atlas on $\mathbf{R}^2$. Show that this atlas is smooth.

3 a)Define an atlas on S^1 with two charts using stereographic coordinates considered in the Homework 1 and show that this atlas is smooth.
b)Do the same for S^2.

4 Define an atlas on $\mathbf{R}P^2$ using inhomogeneous coordinates (see Homework 1). Show that this atlas is smooth. **4b)*** Do the same for $\mathbf{R}P^n$ $(n=2,3,4,\dots)$

5 Define an smooth atlas on $\mathbf{C}P^1$. **5b)*** Do it for $\mathbf{C}P^n$.

6 Is the map $\varphi\colon\ \mathbf{R}\to\mathbf{R}, x\mapsto x^3$ a diffeomorphism?

7 Establish diffeomorphisms between $\mathbf{R}P^1$ and S^1, and between $\mathbf{C}P^1$ and S^2.

7 Show that the special linear group $SL(2)=\left\{g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\colon\ a,b,c,d\in\mathbf{R},\ \det g=1\right\}$ has a natural structure of a differentiable manifold of dimension 3.

8* Show that the special unitary group $SU(2)$

$$SU(2)=\left\{g=\begin{pmatrix}a & b\\ c & d\end{pmatrix} : a,b,c,d\in\mathbf{C}, g^{-1}=g^+, \det g=1\right\}$$

has a natural structure of a differentiable manifold of dimension 3. Show that this manifold is diffeomorphic to S^3. (Recall that g^+ is the matrix which is hermitian conjugate to the matrix g: if $g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$, then $g^+=\begin{pmatrix}\bar{a} & \bar{c}\\ \bar{b} & \bar{d}\end{pmatrix}$)

9* Show that the configuration space of solid body with a fixed point can be identified with the special orthogonal group $SO(3)$. (The group $SO(3)$ is a group of 3×3 real orthogonal matrices with determinant 1, i.e., it is the group of matrices which preserve scalar product and orientation in $\mathbf{R}^3$.)

10† Show that the projective space $\mathbf{R}P^3$ is diffeomorphic to $SO(3)$.