Homework 3.

1 Find transformations of components of velocity vector in $\mathbf{R}^2$ if we change cartesian coordinates on polar coordinates, and vice versa .

2* Let $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ be a basis in the tangent space to $\mathbf{R}^3$ at the given point $\mathbf{x}$ corresponding to the cartesian coordinates (x, y, z) and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi\}$ be a basis at this point corresponding to the spherical coordinates $(x = r\sin\theta\cos\varphi, y = r\sin\theta\sin\varphi, z = r\cos\varphi)$.

Show that $r\mathbf{e}_r = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$.

* Find transformation from the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi\}$.

3 For a tangent space to an arbitrary point on the circle S^1 find the transformation from the basis $\mathbf{e}_\varphi$ (corresponding to the polar angle φ) to the basis $\mathbf{e}_u$ (corresponding to the "stereographic coordinate" u).

4 Consider the vector $2\mathbf{e}_x - \mathbf{e}_y$ at the point $\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$.

a) Show that this vector is tangent to the sphere $x^2 + y^2 + z^2 = 1$.

b) Write down the expression for this vector in spherical coordinates

c) Write down the expression for this vector in stereographic coordinates.

5 Consider a set M in $\mathbf{R}^3$ given by the equation $x^2 + y^2 + z^2 - 4x = 0$. One can see that origin (the point $O = (0, 0, 0)$) belongs to this set. Show that M is a manifold. Find all vectors at the origin which are tangent to this manifold, i.e., belong to the space $T_O M$.

6 A two-dimensional manifold M is specified as a surface in $\mathbf{R}^3$ by the equation $z = f(x, y)$. Considering $x = u, y = v$ as coordinates on M, express the basis vectors $\{\mathbf{e}_u, \mathbf{e}_v\} \in T_\mathbf{x} M$ for an arbitrary point $\mathbf{x} \in M$ as vectors in ambient space $\mathbf{R}^3$.

For manifolds specified in the space of matrices we identify tangent vectors with matrices. The identity matrix is denoted by E.

7 Describe the tangent space $T_E SL(2)$. (See exercise 7 in Homework 2)

8a) For the matrix $A = \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$ calculate $\exp A$ and show that $\exp A \in SL(2)$.

b)*) Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an arbitrary 2×2 matrix. Show that

$$\exp X \in SL(2) \Leftrightarrow \mathrm{TrX} = a + d = 0$$

Recall that matrix exponent $\exp X$ *is defined as the sum of absolutely convergent series* $\exp X = E + X + \frac{X^2}{2} + \frac{X^3}{3!} + \frac{X^4}{4!} + \dots$