Homework 4.

Consider the following vector fields in $\mathbf{R}^3$

$$\mathbf{T}_x = \frac{\partial}{\partial x},\ \mathbf{T}_y = \frac{\partial}{\partial y},\ \mathbf{T}_z = \frac{\partial}{\partial x},\quad \mathbf{L}_x = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y},\quad \mathbf{L}_y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z},\quad \mathbf{L}_z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

1 Calculate $\mathbf{T}_x f$, $\mathbf{L}_x f$, $\mathbf{L}_y f$, $\mathbf{L}_z f$ for $f = x^2 + y^2 + z^2$, $f = x^2 + y^2 - z^2$

2Calculate $(\mathbf{L}_x\mathbf{L}_y - \mathbf{L}_y\mathbf{L}_x)f$ and $\mathbf{L}_z$f for $f = z^n$

3 $\mathbf{L}_x f = \mathbf{L}_y f = 0$. Show that $\mathbf{L}_z f = 0$ also.

4 Show that vector fields $\mathbf{T}_x$, $\mathbf{T}_y$, $\mathbf{T}_z$ commute with each other, i.e. commutators are equal to zero:

$$[\mathbf{T}_x, \mathbf{T}_y] = [\mathbf{T}_y, \mathbf{T}_z] = [\mathbf{T}_z, \mathbf{T}_x] = 0\,.$$

5 Calculate commutators $[\mathbf{L}_x, \mathbf{L}_y]$, $[\mathbf{L}_y, \mathbf{L}_z]$, $[\mathbf{L}_z, \mathbf{L}_x]$.

6 Calculate commutators $[\mathbf{T}_x, \mathbf{L}_x]$, $[\mathbf{T}_x, \mathbf{L}_y]$, $[\mathbf{T}_x, \mathbf{L}_z]$.

7 The linear operator D on smooth functions on $\mathbf{R}^3$ obeys the following conditions:

$$D(fg) = f(\mathbf{x}_0)D(g) + D(f)g(\mathbf{x}_0) \quad \text{for an arbitrary smooth functions } f, g$$

$D(f) = x_0 \cos y_0$ for the function $f = \sin y$
$D(f) = y_0 \sin x_0$ for the function $f = \cos x$
$D(f) = 0$ for the function $f = z$

Show that D is equal to the value of the vector field $\mathbf{L}_z$ at the point $\mathbf{x}_0 = (x_0, y_0, z_0)$.

8* Calculate $\exp(a\mathbf{T}_x)f(x, y, z)$ in the case if $f = f(x, y, z)$ is a polynomial. Is it true that $\exp(a\mathbf{T}_x)f(x, y, z) = f(x + a, y, z)$ for an arbitrary smooth function. (Answer: No!)

9* Calculate $\exp(a\mathbf{L}_z)f(x, y, z)$ (Hint: Use polar (or spherical) coordinates)