Solutions of homework 1

1 *a) Using the stereographic projection from the north pole $N=(0,1)$ introduce stereographic coordinate for the part of the circle S^1 $(x^2+y^2=1)$ without the north pole.*

b) Do the same but using the south pole $S=(0,-1)$ instead of the north pole.

c) express the stereographic coordinates obtained in a) and b) in terms of the angle φ (for polar coordinates in $\mathbf{R}^2$, $x=r\cos\varphi, y=r\sin\varphi$).

a) Consider the line which passes through the north pole $(0,1)$ the point (x,y) on the circle and intersects the axis Ox at the point $(u,0)$; $u=u(x,y)$ is a stereographic coordinate. Three points: north pole, the point (x,y) on the circle and the point $(u,0)$ are on the same line. Hence $\frac{x}{u}+y=1$. On the other hand $x^2+y^2=1$. Substitute $y=1-x/u$ in the equation of the circle we come to quadratic equation:

$$x^2+\left(1-\frac{x}{u}\right)^2=1$$

This equation has two solution $x=0$, $(y=1)$— it is the North pole $(0,1)$ which belongs to the line and the circle. The second solution is $x=\frac{2u}{1+u^2}$. Hence $y=1-x/u=\frac{1-u^2}{1+u^2}$ and $u=\frac{x}{1-y}$. We come to the relations:

$$\begin{cases} x=\frac{2u}{1+u^2} \\ y=\frac{1-u^2}{1+u^2} \end{cases},\quad u=\frac{x}{1-y} \tag{1.1}$$

b) Consider the line which passes through the south pole $(0,-1)$, the point (x,y) on the circle and intersects the axis Ox at the point $(u',0)$. $u'=u(x,y)$ is a stereographic coordinate related with south pole. Under the reflection with respect to the axis Ox this line will go to the line which pass through the south pole $(0,-1)$ and the point $(x,-y)$ on the circle: $x\to x, y\to -y$ in the equation (2).

$$\begin{cases} x=\frac{2u'}{1+(u')^2} \\ y=\frac{(u')^2-1}{1+(u')^2} \end{cases},\quad u'=\frac{x}{1+y} \tag{1.2}$$

The relation between coordinates u and u' is

$$uu'=1\,. \tag{1.3}$$

c) If φ is an angle between axis Ox and the point (x,y) on the circle, then $x=\cos\varphi, y=\sin\varphi$ and

$$u=\frac{x}{1-y}=\frac{2\cos\varphi}{1-\sin\varphi}=\frac{2\sin(\frac{\pi}{2}-\varphi)}{1-\cos\left(\frac{\pi}{2}-\varphi\right)}=2\,\mathrm{cotanh}\left(\frac{\pi}{4}-\frac{\varphi}{2}\right),$$

$$u'=\frac{x}{1+y}=\frac{2\cos\varphi}{1+\sin\varphi}=\frac{2\sin(\frac{\pi}{2}-\varphi)}{1+\cos\left(\frac{\pi}{2}-\varphi\right)}=2\tanh\left(\frac{\pi}{4}-\frac{\varphi}{2}\right)\,. \tag{1.4}$$

2 *a) Using the stereographic projection from the north pole $N=(0,0,1)$ introduce stereographic coordinates (u,v) for the part of the sphere S^2 $(x^2+y^2+z^2=1)$ without the north pole.*

b) Do the same but introduce stereographic coordinates (u',v') using the south pole $S=(0,0,-1)$ instead of the north pole.

c) express these stereographic coordinates via spherical coordinates θ,φ (for spherical coordinates in $\mathbf{R}^3$, $x=r\sin\theta\cos\varphi, y=r\sin\theta\sin\varphi$, $z=r\cos\theta$).

a) In the same way as in the previous exercise consider the line which passes through the North Pole $(0,0,1)$ the point (x,y,z) on the sphere and intersects the plane Oxy at the point with coordinates $(u,v,0)$. $u=u(x,y,z)$, $v=v(x,y,z)$ are stereographic coordinates of the sphere. We come to conditions $x^2+y^2+z^2=1$ and the condition that three points $(0,0,1)$, (x,y,z) and $(u,v,0)$ belong to the same line, i.e. $x:y:z-1=u:v:-1$. We have the system of equations

$$\begin{cases} \frac{x}{y}=\frac{u}{v} \\ \frac{y}{1-z}=v \\ x^2+y^2+z^2=1 \end{cases}$$

Solving these equations we come to

$$\begin{cases} x = \frac{2u}{1+u^2+v^2} \\ y = \frac{2v}{1+u^2+v^2} \\ z = \frac{1-u^2-v^2}{1+u^2+v^2} \end{cases} \text{ and } \begin{cases} u = \frac{x}{1-z} \\ v = \frac{y}{1-z} \end{cases} \tag{2.1}$$

b) Consider the line which passes through the South Pole $(0, 0, -1)$ the point (x, y, z) on the sphere and intersects the plane Oxy at the point with coordinates $(u', v', 0)$. $u' = u'(x, y, z)$, $v' = v'(x, y, z)$ are stereographic coordinates of the sphere related with the South pole. Note that changing $z \to -z$ in the previous formulae lead to these stereographic coordinates related with South Pole $(0, 0, -1)$:

$$\begin{cases} x = \frac{2u'}{1+u'^2+v'^2} \\ y = \frac{2v'}{1+u'^2+v'^2} \\ z = \frac{u'^2+v'^2-1}{u^2+v^2+1} \end{cases} \text{ and } \begin{cases} u' = \frac{x}{1+z} \\ v' = \frac{y}{1+z} \end{cases} \tag{2.2}$$

The relation between u, v and u', v are

$$\begin{cases} \frac{u}{v} = \frac{u'}{v'} \\ uu' + vv' = 1 \end{cases}, \quad \begin{cases} u' = \frac{u}{u^2+v^2} \\ v' = \frac{v}{u^2+v^2} \end{cases} \tag{2.3}$$

Now express stereographic coordinates via spherical coordinates:

$$u = \frac{x}{1-z} = \frac{\sin\theta\cos\varphi}{1-\cos\theta} = 2\,\mathrm{cotan}\frac{\theta}{2}\cos\varphi$$

$$v = \frac{y}{1-z} = \frac{\sin\theta\sin\varphi}{1-\cos\theta} = 2\,\mathrm{cotan}\frac{\theta}{2}\sin\varphi \tag{2.4}$$

i.e.

$$u + iv = 2\,\mathrm{cotan}\frac{\theta}{2}e^{i\varphi}$$

Respectively for coordinates u', v'

$$u' = \frac{x}{1+z} = \frac{\sin\theta\cos\varphi}{1+\cos\theta} = 2\tan\frac{\theta}{2}\cos\varphi$$

$$v = \frac{y}{1+z} = \frac{\sin\theta\sin\varphi}{1+\cos\theta} = 2\tan\frac{\theta}{2}\sin\varphi \tag{2.4}$$

i.e.

$$u + iv = 2\tan\frac{\theta}{2}e^{i\varphi}$$

3* *Introduce stereographic coordinates on n-dimensional sphere* S^n.

Let $x^1, \dots, x^{n+1}$ be cartesian coordinates in $\mathbf{R}^{n+1}$. The point $(0, 0, \dots, 1)$ is the North pole of the sphere $(x^1)^2 + \dots + (x^{n+1})^2 = 1$. The straight line passes through the point of the sphere and the point on the plane $x^{n+1} = 0$ with the coordinates $(u^1, \dots, u^n, 0)$ then $u^1 = u^1(x^1, \dots, x^{n+1}), \dots, u^n = u^n(x^1, \dots, x^{n+1})$ are stereographic coordinates. Performing the analogous calculations we come to the answer:

$$x^k = \frac{2u^k}{1 + (u^1)^2 + (u^2)^2 + \dots + (u^n)^2}, \qquad u^k = \frac{x^k}{1 - x^{n+1}} \qquad \text{for } k = 1, 2, 3, \dots, n;$$

$$x^{n+1} = \frac{1 - (u^1)^2 - (u^2)^2 - \dots - (u^n)^2}{1 + (u^1)^2 + (u^2)^2 + \dots + (u^n)^2} \tag{3.1}$$

Respectively for projective coordinates related with the South Pole we have:

$$x^k = \frac{2u'^k}{1+(u'^1)^2+(u'^2)^2+\ldots+(u'^n)^2}, \qquad u'^k = \frac{x^k}{1+x^{n+1}} \qquad \text{for } k=1,2,3,\ldots,n;$$

$$x^{n+1} = \frac{(u'^1)^2+(u'^2)^2+\ldots+(u'^n)^2-1}{(u'^1)^2+(u'^2)^2+\ldots+(u'^n)^2+1} \tag{3.1}$$

Coordinates $(u^1,\ldots,u^n)$ and $(u^1,\ldots,u^n)$ are related with relations

$$u'^k = \frac{u^k}{1+(u^1)^2+(u^2)^2+\ldots+(u^n)^2} \quad k=1,2,\ldots,n$$

4† *Show that stereographic projection is the bijection between rational points on the unit sphere (the points on the unit sphere with rational coordinates) and rational points on the plane.*

Find all rational solutions of the equations $x^2+y^2=1$, $x^2+y^2+z^2=1$.

Find all integer solutions of the equation $x^2+y^2=z^2$ (Pythagoreans triples.)

The fact that it is bijection between rational points follows from explicit formulae above. On the other hand this is easy to understand without any formula. The relation is given by intersection of straight line with plane and points of the sphere. If coordinates of the sphere are rational then the line has rational coefficients. We come to quadratic equation with rational coefficients. One of its solution is rational, hencethe second is rational too. Hence the line intersects the plane at the point with rational coordinates. ■

The formulae (1.3), (2.3) give solutions: $x,y,z\in\mathbf{Q} \Leftrightarrow u,v,\in\mathbf{Q}$

In particular $x=\frac{2u}{1+u^2}, y=\frac{1-u^2}{1+u^2}$ are rational iff u is rational. Hence taking $u=\frac{p}{q}$ we come to Pythagorian triples $(2p, p^2-q^2, p^2+q^2)$.

5 *Considering the natural bijection of the part of real projective space* [1)] $\mathbf{R}P^n$ *on the plane $x^{n+1}=1$ in $\mathbf{R}^{n+1}$ ($(x^1,\ldots,x^n,x^{n+1})$ are cartesian coordinates on $\mathbf{R}^{n+1}$) one can introduce "inhomogeneous coordinates" on the part of the $\mathbf{R}P^n$.*

a) do this exercise for $n=1,2$; b) do this exercise for an arbitrary n.*

6 *Find different "inhomogeneous" coordinates on $\mathbf{R}P^2$ considering natural bijections of the parts of $\mathbf{R}P^2$ on the plane $x^3=1$ or on the plane $x^2=1$ or on the plane $x^1=1$. Find relations between these coordinates.*

Do the exercises 5 and 6 simultaneously.

Consider in $\mathbf{R}P^n$ $n+1$ the domains $U_1, U_2, \ldots, U_{n+1}$ such that

U_1 is the set of points with homogeneous coordinates $[x^1 : x^2 : \ldots : x^{n+1}]$ such that $x^1 \neq 0$
U_2 is the set of points with homogeneous coordinates $[x^1 : x^2 : \ldots : x^{n+1}]$ such that $x^2 \neq 0$
$\ldots$ U_{n+1} is the set of points with homogeneous coordinates $[x^1 : x^2 : \ldots : x^{n+1}]$ such that $x^{n+1} \neq 0$

We define on every set U_k the map φ_k of U_k on $\mathbf{R}^n$ by dividing the coordinates on the coordinate x_k. Geometrically this means that we find the coordinates of the intersection of the line $[x^1 : x^2 : \ldots : x^{n+1}]$ with the plane $x_{k=1}$ Write down detailed formulae for the cases $n=1,2$

1) $n=1$, $\mathbf{R}P^1$ is the set of equivalence classes $\{[x^1 : x^2]\}$

$$\varphi_1\colon \quad [x^1 : x^2] \mapsto u_{(1)}, \quad u_{(1)} = \frac{x^2}{x^1}, \tag{6.1}$$

[1)] *Real projective space $\mathbf{R}P^n$ = {set of straight lines in $\mathbf{R}^{n+1}$ which pass throug the origin}, or in other words $\mathbf{R}P^n$ is the set of equivalence classes of non-zero vectors in $\mathbf{R}^{n+1}$, i.e. $\mathbf{R}P^n = \{[x^1 : x^2 : \ldots : x^{n+1}\}]$ where $[x^1 : x^2 : \ldots : x^{n+1}]$ stands for the equivalence class of a non-zero vector $(x^1,\ldots,x^{n+1}) \in \mathbf{R}^{n+1}$.—Two non-zero vectors $(x^1,\ldots,x^{n+1})$, $(x'^1,\ldots,x'^{n+1})$ are considered equivalent if they are proportional: $x'^1=\lambda x^1$, $x'^2=\lambda x^2,\ldots,x'^n=\lambda x^n, x'^{n+1}=\lambda x^{n+1}$.*

i.e. the line $[x^1 : x^2]$ intersects the plane $x^1 = 1$ at the point $\left(1, \frac{x^2}{x^1}\right)$.

$$\varphi_2: \quad [x^1 : x^2] \mapsto u_{(2)}, \quad u_{(2)} = \frac{x^1}{x^2} \tag{6.2}$$

i.e. the line $[x^1 : x^2]$ intersects the plane $x^2 = 1$ at the point $\left(\frac{x^1}{x^2}, 1\right)$ We define two charts on $\mathbf{R}P^1$: coordinate $u_{(1)}$ on the domain U_1 ($x_1 \neq 0$), and the coordinate $u_{(2)}$ on the domain U_2 ($x_2 \neq 0$). These charts form the atlas. Transition function is:

$$u_{(2)} = \frac{1}{u_{(1)}}$$

2) $n = 2$, $\mathbf{R}P^n$ is the set of equivalence classes $\{[x^1 : x^2 : x^3]\}$

$$\varphi_1: \quad [x^1 : x^2 : x^3] \mapsto (u^1_{(1)}, u^2_{(1)}), \quad u^1_{(1)} = \frac{x^2}{x^1}, \quad , u^2_{(1)} = \frac{x^3}{x^1}, \tag{6.1}$$

i.e. the line $[x^1 : x^2 : x^3]$ intersects the plane $x^1 = 1$ at the point $\left(1, \frac{x^2}{x^1}, \frac{x^3}{x^1}\right)$.

$$\varphi_2: \quad [x^1 : x^2 : x^3] \mapsto (u^1_{(2)}, u^2_{(2)}), \quad u^1_{(2)} = \frac{x^1}{x^2}, \quad , u^2_{(2)} = \frac{x^3}{x^2}, \tag{6.2}$$

i.e. the line $[x^1 : x^2 : x^3]$ intersects the plane $x^2 = 1$ at the point $\left(\frac{x^1}{x^2}, 1, \frac{x^3}{x^2}\right)$ and

$$\varphi_3: \quad [x^1 : x^2 : x^3] \mapsto (u^1_{(3)}, u^2_{(3)}), \quad u^1_{(3)} = \frac{x^1}{x^3}, \quad , u^2_{(3)} = \frac{x^2}{x^3}, \tag{6.3}$$

i.e. the line $[x^1 : x^2 : x^3]$ intersects the plane $x^3 = 1$ at the point $\left(\frac{x^1}{x^3}, \frac{x^2}{x^3}, 1\right)$.

We define three charts on $\mathbf{R}P^n$: coordinates $(u^1_{(1)}, u^2_{(1)})$ on the domain U_1 ($x_1 \neq 0$), coordinates $(u^1_{(2)}, u^2_{(2)})$ on the domain U_2 ($x_2 \neq 0$) and coordinates $(u^1_{(3)}, u^2_{(3)})$ on the domain U_3 ($x_3 \neq 0$). These charts form the atlas. Transition functions are the following:

$$\text{between } (u^1_{(1)}, u^2_{(1)}) \text{ and } (u^1_{(2)}, u^2_{(2)}): \qquad u^1_{(2)} = \frac{1}{u^1_{(1)}}, \ u^2_{(2)} = \frac{u^2_{(1)}}{u^1_{(1)}}$$

$$\text{between } (u^1_{(1)}, u^2_{(1)}) \text{ and } (u^1_{(3)}, u^2_{(3)}): \qquad u^1_{(3)} = \frac{1}{u^2_{(1)}}, \ u^2_{(3)} = \frac{u^1_{(1)}}{u^2_{(1)}}$$

$$\text{between } (u^1_{(2)}, u^2_{(2)}) \text{ and } (u^1_{(3)}, u^2_{(3)}): \qquad u^1_{(3)} = \frac{u^1_{(2)}}{u^2_{(2)}}, \ u^2_{(3)} = \frac{1}{u^2_{(2)}}$$

7 *Do the previous exercise for complex projective spaces* $\mathbf{C}P^1$ *and* $\mathbf{C}P^2$. Our formulae formally have the same appearance:

1) $\mathbf{C}P^1$ is a set of complex lines $[z^1 : z^2]$ (real planes: $(z^1, z^2) \approx (\lambda z^1, \lambda z^2$ for complex λ). Consider two charts.

First chart: the set U_1 of the equivalence classes $[z^1 : z^2]$ such that $z^1 \neq 0$ and the map

$$\varphi_1: \quad U_1 \to \mathbf{C}: \ u = \frac{z^2}{z^1}, \quad (u \in \mathbf{C})$$

Second chart: the set U_2 of the equivalence classes $[z^1 : z^2]$ such that $z^2 \neq 0$ and the map

$$\varphi_2: \quad U_1 \to \mathbf{C}: \ w = \frac{z^1}{z^2}, \quad (w \in \mathbf{C})$$

We see that $u=\frac{1}{w}$. In terms of real coordinates

$$u=x+iy=\frac{1}{w}=\frac{1}{v+it}, \text{i.e.,}, x=\frac{v}{v^2+t^2}, y=-\frac{t}{v^2+t^2}$$

it is a composition of inversion and reflection with respect to x axis.

2) $\mathbf{C}P^2$ is a set of complex lines $[z^1:z^2:z^3]$ (real planes: $(z^1,z^2)\approx(\lambda z^1,\lambda z^2$ for complex λ). Consider three charts.

First chart: the set U_1 of the equivalence classes $[z^1:z^2:z^3]$ such that $z^1\neq 0$ and the map

$$\varphi_1\colon\quad U_1\to\mathbf{C}^2\colon\ u=\frac{z^2}{z^1}, v=\frac{z^3}{z^1}$$

Second chart: the set U_2 of the equivalence classes $[z^1:z^2:z^3]$ such that $z^2\neq 0$ and the map

$$\varphi_2\colon\quad U_1\to\mathbf{C}^2\colon\ u'=\frac{z^1}{z^2}, v'=\frac{z^3}{z^2}$$

and third chart: the set U_3 of the equivalence classes $[z^1:z^2:z^3]$ such that $z^3\neq 0$ and the map

$$\varphi_3\colon\quad U_1\to\mathbf{C}^2\colon\ \tilde{u}=\frac{z^1}{z^3}, \tilde{v}=\frac{z^2}{z^3}$$

These charts form atlas.