**Solutions of Homework 2**

**1** *Check whether the following subsets are open*

*a) the subset* $\{(x,y)\in\mathbf{R}^2\colon\ x^2+y^2<1\}$,

*b) the subset* $\{(x,y)\in\mathbf{R}^2\colon\ x^2+y^2\leq 1\}$,

*c) the subset* $\{(x,y)\in\mathbf{R}^2\colon\ 0<x<1\}$

*d) the subset* $\{(x,y,z)\in\mathbf{R}^3\colon\ x^2+y^2<1, |z|<1\}$

*e) the subset* $\mathbf{R}^2\backslash I_-$ *, where* $I_-=\{(x,y)\colon\ y=0, x\leq 0\}$.

(a) The domain a) is open, because every point is internal. To check it notice that for an arbitrary point $(x_0,y_0)$ in this domain $(x_0,y_0)\colon x_0^2+y_0^2<1$, there exists $\delta>0$ such that $\sqrt{x_0^2+y_0^2}\leq 1-\delta$. Hence all the points of the ball

$$B_{\frac{\delta}{2}}(x_0,y_0)\colon\qquad \left\{(x,y)|\qquad d((x,y),(x_0.y_0))<\frac{\delta}{2}\right\}$$

belong to this domain. Thus we prove that any point of this domain is internal. Hence the domain is open.

(b) This domain is not open: the points $(x_0,y_0)$ such that $x_0^2+y_0^2=1$ are not internal.

(c) This domain is open, because every point is internal. To check it notice that for an arbitrary point $(x_0,y_0)$ in this domain $(x_0,y_0)\colon 0<x<1$, all the points of the ball $B_\delta(x_0,y_0)$ with $\delta<\frac{1}{2}\min\{|1-x|,|x|\}$ belong to this domain.

(d) The domain is open, because every point is internal. To check it notice that for an arbitrary point $(x_0,y_0,z_0)$ in this domain $(x_0,y_0,z_0)\colon x_0^2+y_0^2<1$, $|z_0|<1$, there exists $\delta>0$ such that $\begin{cases}\sqrt{x_0^2+y_0^2}\leq 1-\delta\\ |z|\leq 1-\delta\end{cases}$. Hence all the points of the ball $B_{\frac{\delta}{2}}(x_0,y_0,z_0)$ belong to this domain. Thus we prove that any point of this domain is internal. Hence the domain is open.

(e) The domain is open. Take an arbitrary point $(x_0,y_0)$ in this domain, i,e, $y_0\neq 0$ or if $y_0=0$ then $x_0>0$. In the first case if $y_0\neq 0$, then there exists $\delta>0$ such that $|y_0|\geq\delta$, hence all the points of the ball $B_{\frac{\delta}{2}}(x_0,y_0)$ belong to the domain.

If $y_0=0$ and $x>0$ then there exist $\delta>0$ such that $x_0\geq\delta$, hence all the points of the ball $B_{\frac{\delta}{2}}(x_0,0)$. belong to this domain. Thus we prove that any point of this domain is internal. Hence the domain is open.

**2** *Consider the sets* $U_1,U_2$ *on* $\mathbf{R}^2$ *such that* $U_1=\mathbf{R}^2\backslash I_-$ *,* $U_2=\mathbf{R}^2\backslash I_+$ *, where* $I_-=\{(x,y)\colon\ y=0, x\leq 0\}$, $I_+=\{(x,y)\colon\ y=0, x\geq 0\}$. *Using polar coordinates define charts* $(U_1,\varphi_1),(U_2,\varphi_2)$. *Show that these charts do not form an atlas on* $\mathbf{R}^2$. *Consider an additional chart* $(U_3,\varphi_3)$, *where* $U_3=\mathbf{R}^2$, $\varphi_3=\mathbf{id}$. *Show that* $\{(U_3,\varphi_3),(U_1,\varphi_1)\}$ *is an atlas on* $\mathbf{R}^2$. *Show that this atlas is smooth.*

The charts $\varphi_1,\varphi_2$ are following:

$$\varphi_1\colon\quad (x,y)\xrightarrow{\varphi_1}(r,\theta_1)\text{ with }-\pi<\theta_1<\pi,\quad \varphi_1\colon\quad (x,y)\xrightarrow{\varphi_1}(r,\theta_2)\text{ with }0<\theta_2<2\pi\left(\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}\right)$$

The map $\varphi_1$ maps the set $U_1$ onto the *open* domain $0<r<\infty, -\pi<\theta_1<\pi$. The map $\varphi_2$ maps the set $U_2$ onto the *open* domain $0<r<\infty, 0<\theta_2<2\pi$. Both maps are bijections on the images. Hence these maps are charts.

The sets $U_1,U_2$ *do not cover the whole* $\mathbf{R}^2$, because the point $(0,0)$ does not belong neither $U_1$, nor $U_2$. Hence these charts do not form an atlas.

The sets $U_2,U_3$ *do cover the whole* $\mathbf{R}^2$, because the points on $I_+$, which do not belong to $U_2$ belong to $U_3$. The transition map

$$\Psi_{32}=\varphi_3\varphi_2^{-1}\colon\quad (r,\varphi)\to(x,y)\colon\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}$$

is smooth: both functions $x=r\cos\theta, y=r\cos\theta$ are smooth.

**3** *a)Define an atlas on $S^1$ with two charts using stereographic coordinates considered in the Homework 1 and show that this atlas is smooth.*
*b)Do the same for $S^2$.*

(a)To solve this problem use the exercise 1 of Homework 1.

Define the first chart $(U_1, \varphi_1)$ using stereographic coordinate related with North pole:

$$U_1 = S^1 \backslash (0,1), \qquad \varphi_1 \colon U_1 \ni (x,y) \mapsto u = \frac{x}{1-y}$$

and

$$U_2 = S^1 \backslash (0,-1), \qquad \varphi_2 \colon U_2 \ni (x,y) \mapsto u' = \frac{x}{1+y}$$

(see solution of exercise 1 in Homework 1) These maps are really charts because in both cases maps are bijections and an image is $\mathbf{R}$—open domain.

The sets $U_1, U_2$ cover the $S^1$. Hence these two charts form an atlas.

The transition function:

$$\Psi_{21} = \varphi_2 \varphi_1^{-1} \colon u \mapsto u' = \frac{1}{u}$$

is smooth function. Hence the atlas is smooth.

(b)To solve this problem use the exercise 2 of Homework 1.

Consider $S^2$ as a set of points $x^2 + y^2 + z^2 = 1$ in $\mathbf{R}^3$.

Define the first chart $(U_1, \varphi_1)$ using stereographic coordinate related with North pole:

$$U_1 = S^1 \backslash (0,0,1), \qquad \varphi_1 \colon U_1 \ni (x,y,z) \mapsto \begin{cases} u = \frac{x}{1-z} \\ v = \frac{y}{1-z} \end{cases}$$

and the second chart $(U_2, \varphi_2)$ using stereographic coordinate related with North pole:

$$U_2 = S^1 \backslash (0,0,-1), \qquad \varphi_2 \colon U_2 \ni (x,y,z) \mapsto \begin{cases} u' = \frac{x}{1+z} \\ v' = \frac{y}{1+z} \end{cases}$$

(see solution of exercise 2 in Homework 1) The image of map $\varphi_1$ and $\varphi_2$ is whole $\mathbf{R}^2$, i.e. image is open. These maps establish one-one correspondence. Thus these maps are really charts.

The sets $U_1, U_2$ cover the $S^1$. Hence these two charts form an atlas. The transition function:

$$\Psi_{21} = \varphi_2 \varphi_1^{-1} \colon (u,v) \mapsto (u',v'), \qquad \begin{cases} u' = \frac{u}{u^2+v^2} \\ v' = \frac{v}{u^2+v^2} \end{cases}$$

(see the exercise 2 in Homework 1) is smooth function. Hence the atlas is smooth.

**4** *Define an atlas on $\mathbf{R}P^2$ using inhomogeneous coordinates (see Homework 1). Show that this atlas is smooth.* **4b)*** *Do the same for $\mathbf{R}P^n$ $(n = 2,3,4,\dots)$*

We use here the exercise 5 of Homework 1. a) First introduce atlas on $\mathbf{R}P^2$.

We consider $\mathbf{R}P^2$ as a set of equivalence classes $[x:y:z]$ of non-zero vectors in $\mathbf{R}^3$.

Define charts $(U_1, \varphi_1)$, $(U_2, \varphi_2)$, $(U_3, \varphi_3)$ such that

$$U_1 = \{[x:y:z] \,\text{such that}\; x \neq 0\} \varphi_1,\; [x:y:z] \mapsto (u_{(1)}, v_{(1)}), \qquad \begin{cases} u_{(1)} = \frac{y}{x} \\ v_{(1)} = \frac{z}{x} \end{cases}$$

$$U_2 = \{[x:y:z] \,\text{such that}\; y \neq 0\} \varphi_2,\; [x:y:z] \mapsto (u_{(2)}, v_{(2)}), \qquad \begin{cases} u_{(2)} = \frac{x}{y} \\ v_{(2)} = \frac{z}{y} \end{cases}$$

$$U_3 = \{[x:y:z]\,\text{such that } z\neq 0\}\varphi_3,\ [x:y:z]\mapsto (u_{(3)},v_{(3)}),\quad \begin{cases} u_{(3)} = \frac{x}{z} \\ v_{(3)} = \frac{y}{z}\end{cases}$$

These three maps are charts because they are bijections on $\mathbf{R}^3$.

These charts form an atlas, because for an arbitrary point $[x:y:z]\in\mathbf{R}P^2$ $x\neq 0$, or $y\neq 0$ or $z\neq 0$.

Transition functions are

$$\Psi_{21} = \varphi_2\varphi_1^{-1},\ (u_{(1)},v_{(1)})\mapsto (u_{(2)},v_{(2)}),\quad \begin{cases} u_{(2)} = \frac{x}{y} = 1:\frac{y}{x} = \frac{1}{u_{(1)}} \\ v_{(2)} = \frac{z}{y} = \frac{z}{x}:\frac{y}{x} = \frac{v_{(1)}}{u_{(2)}}\end{cases}$$

$$\Psi_{32} = \varphi_3\varphi_2^{-1},\ (u_{(2)},v_{(2)})\mapsto (u_{(3)},v_{(3)}),\quad \begin{cases} u_{(3)} = \frac{x}{z} = \frac{x}{y}:\frac{z}{y} = \frac{u_{(2)}}{v_{(2)}} \\ v_{(3)} = \frac{y}{z} = 1:\frac{z}{y} = \frac{1}{v_{(2)}}\end{cases}$$

$$\Psi_{13} = \varphi_1\varphi_3^{-1},\ (u_{(3)},v_{(3)})\mapsto (u_{(1)},v_{(1)}),\quad \begin{cases} u_{(1)} = \frac{y}{x} = \frac{y}{z}:\frac{x}{z} = \frac{v_{(3)}}{u_{(3)}} \\ v_{(1)} = \frac{z}{x} = 1:\frac{x}{z} = \frac{1}{u_{(3)}}\end{cases}$$

All transition functions are smooth, hence the atlas is smooth.

The case of $\mathbf{R}P^n$. We consider $\mathbf{R}P^n$ as a set of equivalence classes $[x^1:x^2:\ldots:x^n:x^{n+1}]$ of non-zero vectors in $\mathbf{R}^{n+1}$.

In this case we have $n+1$-charts $(U_k,\varphi_k)$ (k=1,2,3,…,n+1), where

$$U_k = \{[x^1:x^2:\ldots:x^n:x^{n+1}]\,\text{such that } x^k\neq 0\}$$

and $\varphi_k,\ [x^1:x^2:\ldots:x^n:x^{n+1}]\mapsto (u^1_{(1)},\ldots,u^1_{(1)})$ are

$$\varphi_1\colon\ \begin{cases} u^1_{(1)} = \frac{x^2}{x^1} \\ u^2_{(1)} = \frac{x^3}{x^1} \\ u^3_{(1)} = \frac{x^4}{x^1} \\ \ldots\ldots\ldots \\ {u^k}_{(1)} = \frac{x^{n+1}}{x^1}\end{cases},\quad \varphi_2\colon\ \begin{cases} u^1_{(2)} = \frac{x^1}{x^2} \\ u^2_{(2)} = \frac{x^3}{x^2} \\ u^3_{(2)} = \frac{x^4}{x^2} \\ \ldots\ldots\ldots \\ {u^k}_{(2)} = \frac{x^{n+1}}{x^2}\end{cases},\quad \ldots,\varphi_{n+1}\colon\ \begin{cases} u^1_{(n+1)} = \frac{x^1}{x^{n+1}} \\ u^2_{(n+1)} = \frac{x^2}{x^{n+1}} \\ u^3_{(n+1)} = \frac{x^3}{x^{n+1}} \\ \ldots\ldots\ldots \\ {u^k}_{(n+1)} = \frac{x^n}{x^{n+1}}\end{cases}$$

This collection of charts is an atlas because for an arbitrary point $[x^1:x^2:\ldots:x^n:x^{n+1}]\in\mathbf{R}P^n$ one of components is not equal to zero. One can see that all transition functions are smooth. E.g.

$$\Psi_{21} = \varphi_2\circ\varphi_1^{-1}\ (u^1_{(1)},\ldots,u^1_{(1)})\mapsto (u^1_{(2)},\ldots,u^1_{(2)}),\quad \begin{cases} u^1_{(2)} = \frac{x^1}{x^2} = 1:\frac{x^2}{x^1} = \frac{1}{u^2_{(1)}} \\ u^2_{(2)} = \frac{x^3}{x^2} = \frac{x^3}{x^1}:\frac{x^2}{x^1} = \frac{u^2_{(1)}}{u^1_{(1)}} \\ u^3_{(2)} = \frac{x^4}{x^2} = \frac{x^4}{x^1}:\frac{x^2}{x^1} = \frac{u^3_{(1)}}{u^1_{(1)}}\end{cases}$$

and all components are smooth functions.

**5** *Define an smooth atlas on* $\mathbf{C}P^1$. **5b)*** *Do it for* $\mathbf{C}P^n$.

$\mathbf{C}P^1$ is the set of equivalence classes of pairs of complex numbers $[z^1:z^2]$, such that $z^1\neq 0$ or $z^2\neq 0$.

For $\mathbf{C}P^1$ we have an atlas with two charts:

First chart $(U_1,\varphi_1)$: the set $U_1$ of the equivalence classes $[z^1:z^2]$ such that $z^1\neq 0$ and the map

$$\varphi_1\colon\quad U_1\to\mathbf{C}\colon\ u = \frac{z^2}{z^1},\quad (u\in\mathbf{C})$$

of $U_1$ in $\mathbf{R}^2$ (complex plane)

Second chart: the set $U_2$ of the equivalence classes $[z^1 : z^2]$ such that $z^2 \neq 0$ and the map

$$\varphi_2\colon \quad U_2 \to \mathbf{C}\colon\ w = \frac{z^1}{z^2}, \quad (w \in \mathbf{C})$$

These two maps $\varphi_1, \varphi_2$ are bijections of sphere without a point on the whole plane $\mathbf{R}^2$. Hence they are charts. These two charts form an atlas.

Consider transition function $\Psi_{21} = \varphi_2\varphi_1^{-1}$, $w = \frac{1}{u}$. In terms of real coordinates it will be

$$w = v + it = \frac{1}{u} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}, \quad \begin{cases} v = \frac{x}{x^2+y^2} \\ t = \frac{-y}{x^2+y^2} \end{cases}$$

These both functions are smooth. Hence the atlas is smooth.

In the case $\mathbf{C}P^n$ the atlas formally looks like for the $\mathbf{R}P^n$ but all $x^i$, $u^k_{(r)}$ are complex variables.

**6** *Is the map $\varphi$: $\mathbf{R} \to \mathbf{R}, x \mapsto x^3$ a diffeomorphism?*

This map is one-one map of $\mathbf{R}^1$ on $\mathbf{R}^n$. It is smooth function, but... the inverse function $x \mapsto \sqrt[3]{x}$ is not smooth at the point $x = 0$, since the first derivative already is not defined at this point. Hence this map is not diffeomorphism.

**7** Establish diffeomorphisms between $\mathbf{R}P^1$ and $S^1$, and between $\mathbf{C}P^1$ and $S^2$.

a) Diffeomorphism $\mathbf{R}P^1$ and $S^1$.

Consider $\mathbf{R}P^1$ as set $[x : y]$ of equivalence classes of vectors in $\mathbf{R}^2$ and $S^1$ as set of points $(x, y)\colon x^2 + y^2 = 1$.

The atlas $\{(U_1, \varphi_1), (U_2, \varphi_2)\}$ was already constructed in the exercise 3a:

$$U_1 = S^1 \backslash (0,1),\ \varphi_1\colon U_1 \ni (x,y) \mapsto u = \frac{x}{1-y},\ \ U_2 = S^1 \backslash (0,-1),\ \varphi_2\colon U_2 \ni (x,y) \mapsto u' = \frac{x}{1+y}$$

with transition function

$$\Psi_{21} = \varphi_2\varphi_1^{-1}\colon u \mapsto u' = \frac{1}{u}$$

The atlas on $\mathbf{R}P^1$ is $\{(U_1', \varphi_1'), (U_2', \varphi_2')\}$ where $U_1'$ is a set of points $[x : y]$ with $y \neq 0$, $U_2'$ is a set of points $[x : y]$ with $x \neq 0$,

$$\varphi_1'([x,y] = v = \frac{x}{y},\ \ \varphi_2'([x,y] = v' = \frac{y}{x}$$

with transition function $v' = \frac{1}{v}$.

Define the following map from $S^1$ to $\mathbf{R}P^1$

$$S^1 \ni (x,y), F(x,y) = \begin{cases} [1-y : x] = [1 : \frac{x}{1-y}] \text{ if } y \neq 1 \\ [0 : 1] \text{ if } y = 1 \end{cases}$$

This map establishes one-one correspondence between all the points of the circle except the North Pole and the points of the line $\mathbf{R}^1$ and it sends the North Pole to the "infinity" point $[0 : 1]$ of projective line.

Write down the expression of this map in coordinates:

$$F_{1'1} = \varphi_1'\varphi_1^{-1}\colon u \mapsto v, v = u\,,$$

$$F_{2'1} = \varphi_2'\varphi_1^{-1}\colon u \mapsto v', v' = \frac{1}{u}$$

$$F_{1'2} = \varphi_1'\varphi_2^{-1}\colon u' \mapsto v, v = \frac{1}{u'}\,,$$

$$F_{2'2} = \varphi_2' \varphi_2^{-1} \colon u' \mapsto v', v' = u' \,,$$

All coordinate expressions for the map $F$ is smooth.

We see that the map $F$ is one-one correspondence and in all local charts this function and its inverse are smooth functions. Hence it is diffeomorphism.

**Remark** Notice that one can write the function $F$ in the following "symmetric" way:

$$F(x, y) = \begin{cases} [1 - y : x] \text{ if } y \neq 1 \\ [x : 1 + y] \text{ if } y \neq -1 \end{cases}$$

$([1 - y : x] = [1 - y^2 : x(1 + y)] = [x^2 : x(1 + y)] = [x : 1 + y])$

Now establish diffeomorphism between $S^2$ and $\mathbf{C}P^1$.

We already constructed atlases for these spaces (see exercises 3,5):

Atlas $\{(U_1, \varphi_1), (U_2, \varphi_2)\}$ for $S^2$ is

$$U_1 = S^1 \backslash (0, 0, 1), \qquad \varphi_1 \colon U_1 \ni (x, y, z) \mapsto \begin{cases} u = \frac{x}{1-z} \\ v = \frac{y}{1-z} \end{cases}$$

and

$$U_2 = S^1 \backslash (0, 0, -1), \qquad \varphi_2 \colon U_2 \ni (x, y, z) \mapsto \begin{cases} u' = \frac{x}{1+z} \\ v' = \frac{y}{1+z} \end{cases}$$

with transition function:

$$\Psi_{21} = \varphi_2 \varphi_1^{-1} \colon (u, v) \mapsto (u', v'), \qquad \begin{cases} u' = \frac{u}{u^2+v^2} \\ v' = \frac{v}{u^2+v^2} \end{cases}$$

and atlas $\{(U_1', \varphi_1'), (U_2', \varphi_2')\}$ for $\mathbf{C}P^1$ where

$$\varphi_1' \colon \quad U_1' \to \mathbf{C} \colon \; s = \frac{z^2}{z^1}, \quad (u \in \mathbf{C}) \quad \varphi_2' \colon \quad U_2' \to \mathbf{C} \colon \; s' = \frac{z^1}{z^2}, \quad (w \in \mathbf{C})$$

where $U_1' = \{[z^1 : z^2], z^1 \neq 0\}$, $U_2' = \{[z^1 : z^2], z^2 \neq 0\}$ and transition functions $s = \frac{1}{s'}$ where $s, s'$ are complex variables.

Now consider the following map of $S^1$ on $\mathbf{C}P^1$

$$S^2 \ni (x, y, z), \;\; F(x, y, z) = \begin{cases} [1 - z : x + iy] \text{ if } z \neq 1 \\ [1 + z : x - iy] \text{ if } z \neq -1 \end{cases}$$

This map sends all the points of the sphere except the north pole on the complex plane and North pole to the point $[0 : 1]$. It is one-one correspondence between $S^2$ and $\mathbf{C}P^1$.

Derive expressions in coordinates for the map $F$. We see that $[1 - z : x + iy] = [1 : \frac{x+iy}{1-z}] = [1 : u + iv]$, hence

$$F_{1'1} = \varphi_1' \varphi_1^{-1}, \; s = u + iv$$

and

$$F_{2'1} = \varphi_2' \varphi_1^{-1}, \; s' = \frac{1}{s} = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}$$

Respectively $[1 - z : x + iy] = [(1 - z(x - iy) : x^2 + y^2] = [x - iy : 1 + z] = [\frac{x-iy}{1-z} : 1] = [u' - iv' : 1]$ and

$$F_{2'2} = \varphi_2' \varphi_2^{-1}, \; s' = u' - iv'$$

and

$$F_{1'2} = \varphi_1' \varphi_2^{-1}, \; s = \frac{1}{s'} = \frac{1}{u' - iv'} = \frac{u' + iv'}{u^2 + v^2}$$

We see that the function $F$ is not only bijection but the smooth function, and inverse function is smooth too. Hence $F$ is diffeomorphism.

**7** Show that the special linear group $SL(2) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \ a, b, c, d \in \mathbf{R}, \ \det g = 1 \right\}$ has a natural structure of a differentiable manifold of dimension 3.

Consider $\mathbf{R}^4$ with coordinates $(a, b, c, d)$. The group $SL(2)$ is defined as the set $S$ of the points in $\mathbf{R}^4$ which obeys the equation $F(a, b, c, d) = ad - bc = 1$. Consider derivative matrix of this function:

$$\left( \frac{\partial F(a,b,c,d)}{\partial a}, \frac{\partial F(a,b,c,d)}{\partial b}, \frac{\partial F(a,b,c,d)}{\partial c}, \frac{\partial F(a,b,c,d)}{\partial d} \right) = (d, c, b, a) =$$

This matrix contains one row—vector. Its rank is equal to 1 if this vector is not equal to zero. We have to check that at the points of $S$ where $ad - bc = 1$ the vector $(a, b, c, d) \neq 0$.

At the points where $a \neq 0$ the vector $(a, b, c, d) \neq 0$. At the points of $S$ where $a = 0$ then $cd = 1$ because $ad - dc = 1$. If $cd = 1$ then $c \neq 0$, hence vector $(a, b, c, d) \neq 0$.

We show that at all points of the set $S$ defined by the equation $ad - bc = 1$ the matrix of derivatives has rank 1. Hence according to Theorem $S$ is a smooth manifold of dimension $4 - 1 = 3$.

**8*** *Show that the special unitary group* $SU(2)$

$$SU(2) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbf{C}, g^{-1} = g^{+}, \det g = 1 \right\}$$

*has a natural structure of a differentiable manifold of dimension* 3. *Show that this manifold is diffeomorphic to* $S^3$. *(Recall that* $g^+$ *is the matrix which is hermitian conjugate to the matrix* $g$*: if* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, *then* $g^+ = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$*)*

If matrix $g$ is equal to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and its determinant is equal to 1 then

$$g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \text{ and } g^{+} = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$$

Hence we see that for complex numbers $a, b, c, d$ $d = \bar{a}$, $c = -\bar{b}$. Hence

$$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

where $a, b$ are complex numbers such that $\det g = a\bar{a} + b\bar{b} = 1$. Denote $a = x + iy$, $b = z + it$ we come to

$$\det g = 1 = a\bar{a} + b\bar{b} = (x + iy)(x - iy) + (z + it)(z - it) = x^2 + y^2 + z^2 + t^2$$

We see that $SU(2)$ is a set of points in $\mathbf{R}^4$ obeying the equation $x^2 + y^2 + z^2 + t^2 = 1$. This is $S^3$.

**9*** *Show that the configuration space of solid body with a fixed point can be identified with the special orthogonal group* $SO(3)$. *(The group* $SO(3)$ *is a group of* $3 \times 3$ *real orthogonal matrices with determinant* 1, *i.e., it is the group of matrices which preserve scalar product and orientation in* $\mathbf{R}^3$.*)*

Take any orthogonal frame fixed with body. The position of the body is a linear transformation of this frame. This linear transformation preserves length, hence it preserves scalar product, i.e. it is orthogonal transformation:

$$A^{+} = A$$

If transformation is orthogonal its determinant is equal to 1 (preserving orientation), or $-1$ (changing orientation). Any transformation $\gamma(t)$ can be smoothly connected with identity transformation $\gamma(t)|_{t=0}$. $\det(\gamma(t))$ is continuous function. Hence $\det\gamma(t) = 1$. We see that configuration space can be identified with $SO(3)$

**10**$^\dagger$ *Show that the projective space* $\mathbf{R}P^3$ *is diffeomorphic to* $SO(3)$.

We know that any orthogonal transformation preserving orientation is a rotation on a given angle around a given axis...