Solutions of homework 3.

1 *Find transformations of components of velocity vector in* $\mathbf{R}^2$ *if we change cartesian coordinates on polar coordinates, and vice versa .*

We have

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$$

Hence for derivatives:

$$\mathbf{e}_r = \frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\varphi\mathbf{e}_x + \sin\varphi\ \mathbf{e}_y = \frac{x}{r}\mathbf{e}_x + \frac{y}{r}\mathbf{e}_y$$

$$\mathbf{e}_\varphi = \frac{\partial}{\partial \varphi} = \frac{\partial x}{\partial \varphi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \varphi}\frac{\partial}{\partial y} = -r\sin\varphi\mathbf{e}_x + r\cos\varphi\,\mathbf{e}_y = -y\mathbf{e}_x + x\,\mathbf{e}_y$$

Respectively

$$\mathbf{e}_x = \frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi} = \frac{x}{r}\mathbf{e}_r - \frac{y}{r^2}\mathbf{e}_\varphi$$

$$\mathbf{e}_y = \frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \varphi}{\partial y}\frac{\partial}{\partial \varphi} = \frac{y}{r}\mathbf{e}_r + \frac{x}{r^2}\mathbf{e}_\varphi$$

2* *Let* $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ *be a basis in the tangent space to* $\mathbf{R}^3$ *at the given point* $\mathbf{x}$ *corresponding to the cartesian coordinates* (x, y, z) *and* $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi\}$ *be a basis at this point corresponding to the spherical coordinates* $(x = r\sin\theta\cos\varphi, y = r\sin\theta\sin\varphi, z = r\cos\varphi)$.

Show that $r\mathbf{e}_r = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$.

* *Find transformation from the basis* $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ *to the basis* $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi\}$.

The first exercise is very simple. We have

$$\mathbf{e}_r = \frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} + \frac{\partial z}{\partial r}\frac{\partial}{\partial z} = \frac{x}{r}\mathbf{e}_x + \frac{y}{r}\mathbf{e}_y + \frac{z}{r}\mathbf{e}_z$$

because $r = \sqrt{x^2 + y^2 + z^2}$. Hence

$$r\mathbf{e}_r = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z = r\frac{\partial}{\partial r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$$

To perform transformation from cartesian to spherical coordinates:

$$\mathbf{e}_r = \frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} + \frac{\partial z}{\partial r}\frac{\partial}{\partial z} = \sin\theta\cos\varphi\frac{\partial}{\partial x} + \sin\theta\sin\varphi\frac{\partial}{\partial y} + \cos\theta\frac{\partial}{\partial z}\,,$$

$$\mathbf{e}_\theta = \frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial}{\partial z} = -r\cos\theta\cos\varphi\frac{\partial}{\partial x} + r\cos\theta\sin\varphi\frac{\partial}{\partial y} - r\sin\theta\frac{\partial}{\partial z}$$

$$\mathbf{e}_\varphi = \frac{\partial}{\partial \varphi} = \frac{\partial x}{\partial \varphi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \varphi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \varphi}\frac{\partial}{\partial z} = -r\sin\theta\sin\varphi\frac{\partial}{\partial x} + r\sin\theta\cos\varphi\frac{\partial}{\partial y}$$

3 *For a tangent space to an arbitrary point on the circle S^1 find the transformation from the basis $\mathbf{e}_\varphi$ (corresponding to the polar angle φ) to the basis $\mathbf{e}_u$ (corresponding to the "stereographic coordinate" u).*

See the Coursework solutions

4 *Consider the vector $2\mathbf{e}_x - \mathbf{e}_y$ at the point $\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$.*

a) Show that this vector is tangent to the sphere $x^2 + y^2 + z^2 = 1$.

b) Write down the expression for this vector in spherical coordinates

c) Write down the expression for this vector in stereographic coordinates.

a) The action of this vector on the function $F = x^2 + y^2 + z^2 - 1 = 0$ at the point $\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ is equal to

$$(2F_x - F_y)_{\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)} = 0$$

Hence the vector is tangent to the manifold $F = 0$, sphere.

b) Vector is attached at the point with spherical coordinates $(r, \theta, \varphi,)$, $r = 1$, $\cos\theta = \frac{2}{3}, \tan\varphi = \frac{1}{2}$ $(z = r\cos\theta,\ x = r\sin\theta\sin\varphi,\ x = r\sin\theta\cos\varphi)$ This vector has only x-component and y-component, its z-component is equal to zero. Hence one can see that $\mathbf{e}_\theta$-component will be equal to zero.

From the previous exercise we see that this ∂_r component of this vector is equal to zero to because this vector is tangent to the sphere. Hence this vector has only $\partial\varphi$ component. It is equal to the length of this vector divided to the length of the vector $\mathbf{e}_\varphi$.

5 *Consider a set M in $\mathbf{R}^3$ given by the equation $x^2 + y^2 + z^2 - 4x = 0$. One can see that origin (the point $O = (0, 0, 0)$) belongs to this set. Show that M is a manifold. Find all vectors at the origin which are tangent to this manifold, i.e., belong to the space $T_O M$.*

For the function $F(x, y, z) = x^2 + y^2 + z^2 - 4x$ consider vector derivatives

$$\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) = (2x - 4, 2y, 2z)$$

We see that this vector does not vanish at zeros of the function F. Indeed if $x = 2$, $y = z = 0$, that is vector of derivatives vanishes then $F \neq 0$. Thus all the points where $F = 0$ are regular. Vector of derivatives is not equal to zero (Formally speaking matrix of derivatives has rank 1) and $F = 0$ is smooth manifold dimension 1.

6 *A two-dimensional manifold M is specified as a surface in $\mathbf{R}^3$ by the equation $z = f(x, y)$. Considering $x = u, y = v$ as coordinates on M, express the basis vectors $\{\mathbf{e}_u, \mathbf{e}_v\} \in T_\mathbf{x} M$ for an arbitrary point $\mathbf{x} \in M$ as vectors in ambient space $\mathbf{R}^3$.*

We have for a surface: $x = \mathbf{u}$, $y = \mathbf{v}$, $z = f(\mathbf{u}, \mathbf{v})$.

$$\mathbf{e}_\mathbf{u} = \frac{\partial}{\partial \mathbf{u}} = \frac{\partial x}{\partial \mathbf{u}}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \mathbf{u}}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \mathbf{u}}\frac{\partial}{\partial z} = \mathbf{e}_x + f_x \mathbf{e}_z$$

$$\mathbf{e}_\mathbf{v} = \frac{\partial}{\partial \mathbf{v}} = \frac{\partial x}{\partial \mathbf{v}}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \mathbf{v}}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \mathbf{v}}\frac{\partial}{\partial z} = \mathbf{e}_y + f_y \mathbf{e}_z$$

For manifolds specified in the space of matrices we identify tangent vectors with matrices. The identity matrix is denoted by E.

7 *Describe the tangent space $T_E SL(2)$. (See exercise 7 in Homework 2)*

8a) *For the matrix* $A = \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$ *calculate* $\exp A$ *and show that* $\exp A \in SL(2)$.

b)) Let* $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be an arbitrary* 2×2 *matrix. Show that*

$$\exp X \in SL(2) \Leftrightarrow \mathrm{TrX} = a + d = 0$$

Recall that matrix exponent $\exp X$ *is defined as the sum of absolutely convergent series* $\exp X = E + X + \frac{X^2}{2} + \frac{X^3}{3!} + \frac{X^4}{4!} + \dots$

(Last two exercises see solutions in solutions of Coursework)