Solution of Homework 4.

Consider the following vector fields in $\mathbf{R}^3$

$$\mathbf{T}_x = \frac{\partial}{\partial x},\ \mathbf{T}_y = \frac{\partial}{\partial y},\ \mathbf{T}_z = \frac{\partial}{\partial x}, \quad \mathbf{L}_x = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad \mathbf{L}_y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \quad \mathbf{L}_z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

1 *Calculate* $\mathbf{T}_x f$, $\mathbf{L}_x f$, $\mathbf{L}_y f$, $\mathbf{L}_z f$ *for* $f = x^2 + y^2 + z^2$, $f = x^2 + y^2 - z^2$

$\mathbf{T}_x f = \frac{\partial f}{\partial x} = 2x$, $\mathbf{L}_x f = y f_z - z f_y = 0$.

2 *Calculate* $(\mathbf{L}_x \mathbf{L}_y - \mathbf{L}_y \mathbf{L}_x) f$ *and* $\mathbf{L}_z f$ *for* $f = z^n$

3 $\mathbf{L}_x f = \mathbf{L}_y f = 0$. *Show that* $\mathbf{L}_z f = 0$ *also.*

One can perform exercise 2 just by straightforward calculations using "brut force". We prefer imply it from exercise 3.

$$[\mathbf{L}_x, \mathbf{L}_y] = \mathbf{L}_x \mathbf{L}_y - \mathbf{L}_y \mathbf{L}_x = (y\partial_z - z\partial_y)(z\partial_x - x\partial_z) - (z\partial_x - x\partial_z)(y\partial_z - z\partial_y) =$$

$$y\partial_x - x\partial_y = -\mathbf{L}_z$$

Now it is easy to perform exercise 2. It is evident that $L_z z^n = 0$ because operator L_z does not possess derivation with respect to z.

4 *Show that vector fields* $\mathbf{T}_x$, $\mathbf{T}_y$, $\mathbf{T}_z$ *commute with each other, i.e. commutators are equal to zero:*

$$[\mathbf{T}_x, \mathbf{T}_y] = [\mathbf{T}_y, \mathbf{T}_z] = [\mathbf{T}_z, \mathbf{T}_x] = 0\,.$$

because coeficients in fronts of derivatives $\partial_x, \partial_y, \partial_z$ are constants.

5 *Calculate commutators of vector fields* $[\mathbf{L}_x, \mathbf{L}_y]$, $[\mathbf{L}_y, \mathbf{L}_z]$, $[\mathbf{L}_z, \mathbf{L}_x]$.

6 *Calculate commutators* $[\mathbf{T}_x, \mathbf{L}_x]$, $[\mathbf{T}_x, \mathbf{L}_y]$, $[\mathbf{T}_x, \mathbf{L}_z]$.

Operators $\mathbf{T}_x, \mathbf{T}_y, \mathbf{T}_z$ evidently commute: $[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}] = 0$

As in exercise exercise 3 for commutators

$$[\mathbf{L}_x, \mathbf{L}_y] = -\mathbf{L}_z, \qquad [\mathbf{L}_y, \mathbf{L}_z] = -\mathbf{L}_x, \qquad [\mathbf{L}_z, \mathbf{L}_x] = -\mathbf{L}_y$$

7 *The linear operator* D *on smooth functions on* $\mathbf{R}^3$ *obeys the following conditions:*

$$D(fg) = f(\mathbf{x}_0)D(g) + D(f)g(\mathbf{x}_0) \quad \textit{for an arbitrary smooth functions } f, g$$

$D(f) = x_0 \cos y_0$ *for the function* $f = \sin y$

$D(f) = y_0 \sin x_0$ *for the function* $f = \cos x$

$D(f) = 0$ *for the function* $f = z$

Show that D is equal to the value of the vector field $\mathbf{L}_z$ at the point $\mathbf{x}_0 = (x_0, y_0, z_0)$.

It follows from the properties of the operator D that D is the the tangent vector at the given point:

$$D = (a\partial_x + b\partial_y + c\partial_z)\big|_{x_0, y_0, z_0} .$$

Applying an operator D to the function $f = \sin y$ we come to the condition that $b = x_0$
Applying an operator D to the function $f = \cos x$ we come to the condition that $a = -y_0$
Applying an operator D to the function $f = z$ we come to the condition that $c = 0$

We see that

$$D = x_0\partial_y - y_0\partial_y$$

i.e. at the point (x_0, y_0, z_0) operator D coincides with operator $\mathbf{L_z}$

8* *Calculate* $\exp(a\mathbf{T}_x)f(x, y, z)$ *in the case if $f = f(x, y, z)$ is a polynomial. Is it true that* $\exp(a\mathbf{T}_x)f(x, y, z) = f(x + a, y, z)$ *for an arbitrary smooth function. (Answer: No!)*

9* *Calculate* $\exp(a\mathbf{L}_z)f(x, y, z)$ *(Hint: Use polar (or spherical) coordinates)*

In the case if Taylor series converges to the function in the vicinity of the point (i.e. function is analytical) the following formal calculations is not meaningless (of course for polynomials it is always meaningful):

$$e^{a\frac{d}{dx}} f = \sum \frac{1}{n!} a^n \frac{d^n}{dx^n} f = f(x + a)$$