Homework 5. Solutions

1 *Consider function $f = xy$ and differential forms $\sigma = xdy + ydx$ and $\omega = zdx + xdy$. Calculate differential forms $d(f\omega)$, $\sigma \wedge \omega$ and $d(\sigma \wedge \omega)$.*

$d(f\omega) = df\omega + fd\omega = d(xy)(zdx + xdy) + xyd(zdx + xdy) = (ydx + xdy) \wedge (zdx + xdy) + xy(dz \wedge dx + dx \wedge dy) = xzdy \wedge dx + yxdx \wedge dy + xydz \wedge dx + xydx \wedge dy =$

$= (2xy - xz)dx \wedge dy + xydz \wedge dx.$

$\sigma \wedge \omega = (xdy + ydx) \wedge (zdx + xdy) = xzdy \wedge dx + yxdx \wedge dy = (xy - xz)dx \wedge dy$

$d(\sigma \wedge \omega) = d((xy - xz)dx \wedge dy) = -xdz \wedge dx \wedge dy$

Note that $\sigma = df$, hence $d(\sigma \wedge \omega) = d(df \wedge \omega)) = -df \wedge dw$. Here this does not simplify much calculations but in the next exercise it will help.

2 *Consider embedding $\iota: S^2 \to \mathbf{R}^3$ of the sphere to $\mathbf{R}^3$ given by the equation $x = a\sin\theta\cos\varphi$, $y = a\sin\theta\sin\varphi$, $z = a\cos\theta$. Calculate pull-backs*

a) $\iota^*(f)$,

b) $\iota^*(\sigma)$,

c) $\iota^*(\omega)$,

d) $\iota^*(\sigma \wedge \omega)$,

e) $\iota^*(d(\sigma \wedge \omega))$

where function f and forms ω and σ were defined in previous exercise.

We have that $x = a\sin\theta\cos\varphi$, $y = a\sin\theta\sin\varphi$, $z = a\cos\theta$. Hence

$$\iota^*(f) = \iota^*(xy) = (a\sin\theta\cos\varphi)(a\sin\theta\sin\varphi) = a^2\sin^2\theta\sin\varphi\cos\varphi$$

b)

$$\iota^*(\sigma) = \iota^*(xdy + ydx) = a\sin\theta\cos\varphi d(a\sin\theta\sin\varphi) + a\sin\theta\sin\varphi d(a\sin\theta\cos\varphi) =$$

$$a\sin\theta\cos\varphi(a\cos\theta\sin\varphi d\theta + a\sin\theta\cos\varphi d\varphi) + a\sin\theta\sin\varphi d(a\cos\theta\cos\varphi d\theta - a\sin\theta\sin\varphi d\varphi)$$

Long calculations... More wise to note that $\sigma = xdy + ydx = d(xy) = df$, hence

$$\iota^*(\sigma) = \iota^*(df) = d\iota^*(f)$$

But we already calculated $\iota^*(f) = a^2\sin^2\theta\sin\varphi\cos\varphi$. Hence

$$\iota^*(\sigma) = \iota^*(df) = d\iota^*(f) = d\left(a^2\sin^2\theta\sin\varphi\cos\varphi\right) = d\left(a^2\sin^2\theta\sin\varphi\cos\varphi\right) =$$

$$a^2\frac{d((1-\cos 2\theta)\sin 2\varphi}{4} = a^2\frac{\sin 2\theta\sin 2\varphi}{2}d\theta + a^2\frac{((1-\cos 2\theta)\cos 2\varphi}{2}d\varphi$$

c)

$$\iota^*(\omega) = \iota^*(zdx + xdy)$$

$z = a\cos\theta$, $x = a\sin\theta\cos\varphi$, $dx = d(a\sin\theta\cos\varphi)$, $dy = d(\sin\theta\sin\varphi)$. Hence

$$\iota^*(\omega) = \iota^*(zdx + xdy) = a\cos\theta d(a\sin\theta\cos\varphi) + a\sin\theta\cos\varphi d(a\sin\theta\sin\varphi)$$

d) $\iota^*(\sigma\wedge\omega) = \iota^*((xy - xz)dx\wedge dy) =$

$$= (a\sin\theta\cos\varphi\ a\sin\theta\sin\varphi - a\sin\theta\cos\varphi\ a\cos\theta)d(a\sin\theta\cos\varphi)\wedge d(a\sin\theta\sin\varphi) =$$

$$a^4\sin\theta\cos\varphi(\sin\theta\sin\varphi - \cos\theta)(\cos\theta\cos\varphi d\theta - \sin\theta\sin\varphi d\varphi)\wedge(\cos\theta\sin\varphi d\theta + \sin\theta\cos\varphi d\varphi) =$$

$$a^4\sin\theta\cos\varphi(\sin\theta\sin\varphi - \cos\theta)\sin\theta\cos\theta d\theta\wedge d\varphi$$

e) $d(\sigma\wedge\omega)$ is 3-form.

Hence $\iota^*\left(d(\sigma\wedge\omega)\right)$ as a form on is equal to zero. Answer: $\iota^*\left(d(\sigma\wedge\omega)\right) = 0$.

3 *Consider the embedding* $\iota\colon M \to \mathbf{R}^2$ *of the circle* S^1 *in* $\mathbf{R}^2$ *given by the equation* $x = a\cos\theta, y = a\sin\theta$. *Find the pull-backs* $\iota^*(\sigma)$ *and* $\iota^*(d\sigma)$ *if* $\sigma = \frac{xdy - ydx}{x^2+y^2}$.

$$\iota^*\sigma = \frac{a\cos\theta d(a\sin\theta) - a\sin\theta d(a\cos\theta)}{(a\cos\theta)^2 + (a\cos\theta)^2} = \frac{(a^2\cos^2\theta + a^2\sin^2\theta)d\theta}{a^2\cos^2\theta + a^2\sin^2\theta} = d\theta$$

$$\iota^*(d\sigma) = d(\iota^*(\sigma)) = d(d\theta) = 0$$

4 *Consider the embedding* $\iota\colon M \to \mathbf{R}^3$ *of the cylinder* M *in* $\mathbf{R}^3$ *given by the equation* $x = a\cos\theta, y = a\sin\theta, z = h$. *Find the pull-backs* $\iota^*(\sigma)$ *and* $\iota^*(d\sigma)$ *for the following forms:*

$\sigma = zdy$

$\sigma = xdy + ydx$

$\sigma = \frac{xdy - ydx}{x^2+y^2}$,

$\sigma = xdy - ydx$

$$\iota^*(zdy) = hd(a\sin\theta) = ha\cos\theta d\theta$$

$$\iota^*(xdy + ydx) = \iota^* d(xy) = d(\iota^*(xy)) = d(a^2\sin\theta\cos\theta) = a^2\cos 2\theta d\theta$$

$$\iota^*\frac{xdy - ydx}{x^2+y^2} = \frac{a\cos\theta d(a\sin\theta) - a\sin\theta d(a\cos\theta)}{(a\cos\theta)^2 + (a\cos\theta)^2}$$

$$\iota^*(xdy - ydx) = a^2 d\theta$$

5* *Show that* $\iota^*(\sigma) = \sin\theta d\theta\wedge d\varphi$, *where* ι *is embedding of the sphere in* $\mathbf{R}^3$ *considered in the exercise 2, and the 2-form* σ *is defined by the formula*

$$\sigma = \frac{xdy\wedge dz + ydz\wedge dx + zdx\wedge dy}{(x^2+y^2+z^2)^{3/2}}$$

Perform calculations

$$\iota^*(xdy \wedge dz) = a^3 \sin\theta \cos\varphi d(\sin\theta \sin\varphi) \wedge d(\cos\theta) = a^3 \sin^3\theta \cos^2\varphi d\theta \wedge d\varphi$$

$$\iota^*(ydz \wedge dx) = a^3 \sin\theta \sin\varphi d(\cos\theta) \wedge d(\sin\theta \cos\varphi) = \sin^3\theta \sin^2\varphi d\theta \wedge d\varphi$$

$$\iota^*(zdx \wedge dy) = a^3 \cos\theta d(\sin\theta \cos\varphi) \wedge d(\sin\theta \sin\varphi) = a^3 \sin\theta \cos^2\theta d\theta \wedge d\varphi$$

Hence

$$\iota^*\left(\frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2+y^2+z^2)^{3/2}}\right) = \frac{\iota^*(xdy \wedge dz) + \iota^*(ydz \wedge dx) + \iota^*(zdx \wedge dy)}{\iota^*(x^2+y^2+y^2))} =$$

$$\frac{a^3 \sin^3\theta \cos^2\varphi d\theta \wedge d\varphi + a^3 \sin^3\theta \sin^2\varphi d\theta \wedge d\varphi + a^3 \sin\theta \cos^2\theta d\theta \wedge d\varphi}{a^3} = \sin\theta d\theta \wedge d\varphi$$

6 *Calculate differential of* 1*-form* $\alpha = p_1 dq^1 + p_2 dq^2 - dp_2 q^2$

$$d\alpha = d(p_1 dq^1 + p_2 dq^2 - dp_2 q^2) = dp_1 \wedge dq^1 + dp_2 \wedge dq^2 + dp_2 \wedge dq^2$$

(Note that $d(dp_2 f) = -dp_2 df$)

7 *Consider in* $\mathbf{R}^2$ *a triangle* $\triangle ABC$ *with vertices at the points* $A = (5, -1)$, $B = (-1, 6)$, $C = (-5, -1)$ *and differential one-form* $\omega = xdy - ydx$. *By using Stokes' theorem or directly calculate the integral of* 1*-form* ω *over the boundary of the* $\triangle ABC$.

The integral of the 1-form $\omega = xdy - ydx$ over the boundary of the triangle according to Stokes theorem is equal to the integral of the form $d\omega = d(xdy - ydx) = dx \wedge dy - dy \wedge dx = 2dx \wedge dy$ over the triangle:

$$\int_{\partial\triangle ABC} \omega = \int_{\triangle ABC} d\omega$$

(We suppose that orientation is chosen. The integral on boundary is taken anticlockwise.) Hence

$$\int_{\triangle ABC} d\omega = \int_{\triangle ABC} 2dx \wedge dy = 2 \times \text{Area of triangle } ABC = 2 \times (5 - (-5))(6 - (-1)) \times \frac{1}{2} = 70 \blacksquare$$

8 *Consider in* $\mathbf{R}^3$ *the surface defined by the equation* $x^2 + y^2 + z^2 = 4z$. *Show that this surface is the sphere.*

Using Stokes Theorem calculate the integrals of the 2*-forms* $\omega_1 = zdx \wedge dy$ *and* $\omega_2 = dx \wedge dy$

$x^2 + y^2 + z^2 - 4z = 0$. Hence $x^2 + y^2 + (z-2)^2 = 4$. Hence it is the sphere of the radius 2 with the centre at the point $(0,0,2)$, $\int_{S^2} z dx \wedge dy = \int_{\partial B} z dx \wedge dy$, where B is a ball of the radius 2, $B \colon x^2 + y^2 + (z-2)^2 \leq 4$. Hence by Stokes Theorem

$$\int_{S^2} zdx{\wedge}dy = \int_{\partial B} zdx{\wedge}dy = \int_B d(zdx{\wedge}dy) = \int_B dz{\wedge}dx{\wedge}dy = \int_B dx{\wedge}dy{\wedge}dz = volume B =$$

$$\text{volume of the sphere of the radius } 2 = \frac{4}{3}\pi R^3|_{R=2}$$