Homework 6. Solutions

1 *Calculate $H^0_{DR}(M)$ in the case if manifold M is*

a) $M = \mathbf{R}$

Answer: $H^0_{DR}(\mathbf{R}) = \mathbf{R}$

b) $M = S^1$,

Answer: $H^0_{DR}(\mathbf{R}) = \mathbf{R}$

c) M=$\mathbf{R}^n$,

Answer: $H^0_{DR}(\mathbf{R}) = \mathbf{R}$

d) $M = \mathbf{R}^n \backslash \{0\}$ $(n = 1, 2, 3, \dots)$,

Answer: $H^0_{DR}(\mathbf{R}) = \mathbf{R}$ if $n \geq 2$ and $H^0_{DR}(\mathbf{R}) = \mathbf{R}^2$ if $n = 1$.

e) *M is an arbitrary topological manifold.*

Answer: $H^0_{DR}(\mathbf{R}) = \mathbf{R}^n$, where
n is number of connected components.

Explanations to answers:

As we know from the lecture course for any manifold M, the dimension of the space $H^0(M)$ is the number of connected components of M (if it is finite).

Let us recall that the space of 0-forms is the space of smooth functions on M. A function $f \in \Omega^0(M)$ is closed if $df = 0$. What does it mean? Near each point f must be a constant (indeed, we may introduce coordinates and write $df = 0$ in coordinates). Hence f is a *local constant*. It need not be a constant on the whole M (a global constant), which is demonstrated by the example of a manifold consisting of two connected components such as two disjoint copies of $\mathbf{R}^n$. The function may be zero on one component and 1, on another. Notice that there no exact 0-forms (because there are no -1-forms). Therefore $H^0(M)$ is the space of all local constants on M. It is intuitively clear and can be proved by a simple topological argument that on a connected topological space any local constant is a constant. Therefore, if a topological space, in particular, a manifold M, is the disjoint union of its connected components (maximal connected subspaces), then any local constant is defined by its values of the components (constants); thus it is a function on the set of components. If the number of components is finite, the space of functions on this set is finite-dimensional. As a basis one take functions that are identically 1 on one component and identically 0 on all other components. Hence the dimension is the number of components.

2 *Calculate all de Rham cohomology groups $H^k_{DR}(M)$ for*

a) $M = \mathbf{R}^1$,

b) $M = S^1$.

As we know from the exercise 1) $H^0_{DR}(\mathbf{R}) = \mathbf{R}^1$. An arbitrary 1-form $f dx$ on $\mathbf{R}$ is closed and it is exact: $f dx = dF$, because an arbitrary function has antiderivative. Hence $H^1_{DR}(\mathbf{R}) = 0$.

For S^1 again $H^0_{DR}(S^1) = \mathbf{R}^1$ because S^1 is connected. An arbitrary 1-form on $f d\varphi$ on S^1 is closed. Is it right that any closed form is exact? Of course no! E.g.one-form $d\varphi$ is not exact because $\int_{S^1} d\varphi = 2\pi \neq 0$. If form ω is exact $\omega = dr$ then due to Stokes theorem

$$\int_{S^1} \omega = \int_{S^1} dr = \int_{\partial S^1} r = 0$$

Hence we see that form $d\varphi$ is not homologous to zero $[d\varphi] \neq 0$ in $H^1_{DR}(M)$. On other hand one can see that cohomology classes of arbitrary forms are proportional to $[d\varphi]$. Indeed let ω be an arbitrary one-form and $\int_{S^1} \omega = Q$. Consider one form

$$\omega' = \omega - \frac{Q}{2\pi} d\varphi$$

One can see that

$$\int_{S^1} \omega' = \int_{S^1} \omega - \frac{Q}{2\pi} \int_{S^1} d\varphi = 0$$

One can easy to see that the condition that $\int_{S^1} \omega' = 0$ is enough that ω' is exact form. (The antiderivative exists.). Hence we see that for any form w, $[w] = \frac{Q}{2\pi}[d\varphi]$. This means that cohomology group is one-dimensional.

Thus we prove that $H^1(S^1) = \mathbf{R}$.

Compare answers.

3 *Calculate all de Rham cohomology groups $H^k_{DR}(M)$ for manifolds*

a) $N = S^1$,

b) $M = \mathbf{R}^2 \backslash \{0\}$.

Compare answers

We already calculated cohomology groups for S^1: $H^0_{DR}(S^1) = \mathbf{R}, H^1_{DR}(S^1) = \mathbf{R}$.

(All higher groups are zero because dimension of S^1 is 1).

Calculate $H^0_{DR}, H^1_{DR}, H^2_{DR}$. (All higher groups are equal to zero) $H^0_{DR}(\mathbf{R}^2 \backslash \{0\}) = \mathbf{R}$ because $\mathbf{R}^2 \backslash \{0\}$ is connected.

One can see that 1-form $\sigma = d\varphi = \frac{xdy - ydx}{x^2+y^2}$ is closed but not exact form because $\int_{x^2+y^2=1} \sigma = 2\pi \neq 0$. By the same arguments that for circle one can see that $H^1(\mathbf{R}^2 \backslash \{0\}) = \mathbf{R}$.

Obviously any two form $\omega = a(x,y) dx \wedge dy$ for 2-dimensional manifold is closed, because it is top degree form. (All k forms for 2-dimensional manifold vanish if $k \geq 3$.) One can show that it is exact and $H^2(\mathbf{R}^2 \backslash \{0\}) = 0$. To show it in more economical way perform calculations in polar coordiantes. $\omega = \omega = a(x,y) dx \wedge dy = a(r,\theta) r dr \wedge d\theta$ ($dx \wedge dy = r dr \wedge d\theta$), where $a(r,\varphi)$ is defined for $0 < r < \infty$. We see that one can find one form $b(r,\theta) d\theta$ such that

$$a(r,\theta) r dr \wedge d\theta = d(b(r,\theta) d\theta)\,, \text{i.e. } a(r,\theta) = \frac{1}{r} \frac{\partial b(r,\theta)}{\partial r}$$

E.g. $b(r,\theta)=\int_1^r ta(t,\theta)dt$, i.e. the form ω is exact.

We see that cohomology groups for $\mathbf{R}^2\backslash\{0\}$ are the same as for S^1. This follows from the fact these spaces are it homotopy equivalent.—

4 *Consider the space $M=\mathbf{R}^2\backslash\{0\}$. Show that it is not a star-shaped domain. Consider the form*

$$\omega=\frac{xdy-ydx}{x^2+y^2}\,. \tag{1}$$

Calculate the integral of this 1-form over
a) a unit circle S^1 $(x^2+y^2=1)$,
b) over closed curve which is a boundary of domain which contains an origin,
c) over a closed curve which is a boundary of domain which does not contain an origin.

If it is the star shaped domain with respect the point $\mathbf{x}_0$ then consider the segment which passes through the point 0 we come to contradiction.

It is evident that $\iota*\omega=d\varphi$ for embedding ι of S^1 in $\mathbf{R}^2$ by $x=\cos\theta$, $y=\sin\theta$:

$$\iota^*\omega=\iota^*\left(\frac{xdy-ydx}{x^2+y^2}\right)=\frac{\cos\theta d(\sin\theta)-\sin\theta d(\cos\theta)}{(\cos\theta)^2+(\sin\theta)^2}=\frac{\cos^2\theta d\theta+\sin^2\theta d\theta}{\cos^2\theta)+\sin^2\theta}=d\theta$$

Hence

$$\int_{\iota(S^1)}\omega=\int_{S^1}\iota^*\omega=\int_{S^1}d\theta=2\pi$$

(Sometimes $\int_{\iota(S^1)}\omega$ we write just as $\int_{S^1}\omega$)

Let C be an arbitrary closed curve which is a boundary of domain D which contains an origin and let S^1 be a circle with a radius r $(x^2+y^2=r^2)$where we choose r such that S^1 belongs to the domain D. Consider the domain $\tilde{D}=D\backslash B$ where $B=\{x,y,x^2+y^2<r^2\}$

The boundary of domain $\tilde{D}$ is the closed curve C and the closed curve S^1. The form $\omega=\omega=\frac{xdy-ydx}{x^2+y^2}$ is closed form in $\mathbf{R}^2\backslash 0$ (because in polar coordiantes it is equal to $d\theta$). Hence we can apply Stokes Theorem for the domain $\tilde{D}$:

$$\int_{\partial\tilde{D}}\omega=\int_{\tilde{D}}d\omega=0$$

On the other hand:

$$\int_{\partial\tilde{D}}\omega=\int_C\omega-\int_{S^1}\omega\,.$$

Hence $\int_C\omega=\int_S^1\omega=2\pi$ (The sign "-" comes because of orientation.)

In the case if C an closed curve which is a boundary of domain D which does not contain an origin then one can apply immediately Stokes Theorem to the domain D ($\partial D=C$) because the form ω is regular closed form in this domain:

$$\int_C\omega=\int_{\partial D}\omega=\int_D d\omega=0$$

since $d\omega = 0$

5* Let σ be an arbitrary closed 1-form on $M = \mathbf{R}^2 \backslash \{0\}$ and an integral of this form over the unit circle $x^2 + y^2 = 1$ is equal to q. Consider the form $\sigma' = \sigma - \frac{q}{2\pi}\omega$, where ω is the form defined by the equation (1). Show that this an exact form

Hint (solution????): One can see that an integral of this form over the unit circle $x^2 + y^2 = 1$ is equal to 0, hence consider the scalar function $\Phi(x)$ which is equal to the integral of σ' over the curve finishing at the point x. One can see that $d\Phi = \sigma'$

6 a) *Calculate de Rham cohomology groups for $\mathbf{R}^2$.*

b) Show an example of closed two form non-homologous to zero on S^2. Calculate de Rham cohomology groups for S^2. Explain why $H^2_{DR}(S^2 \backslash N) = 0$.

One can see that $H^0_{DR}(\mathbf{R}^2) = \mathbf{R}$ (See for example exercise 1).

It follows from Poincare lemma that $H^1_{DR}(\mathbf{R}^2) = 0, H^2_{DR}(\mathbf{R}^2) = 0$ Perform explicit calculations.

If $\omega = adx + bdy$ is closed form, then to find Φ such that $d\Phi = adx + bdy$ one can perform the same calculations that we did in the lecture course:

$$\Phi(x,y) = x \int_0^1 a(tx,ty)dt + y \int_0^1 b(tx,ty)dt$$

(It is really very good exercise to try to check again that $\frac{\partial \Phi(x,y)}{\partial x} = a(x,y)$ and $\frac{\partial \Phi(x,y)}{\partial y} = b(x,y)$ (of course under assumption that $d\omega = 0$, i.e. $\frac{\partial a(x,y)}{\partial y} = \frac{\partial b(x,y)}{\partial x}$)

So we check straightforwardly that an arbitrary closed 1-form in $\mathbf{R}^2$ is exact.

Now show that an arbitrary closed 2-form in $\mathbf{R}^2$ is exact.

Let $a(x,y)dx \wedge dy$ be an arbitrary 2-form. It is closed, because it has maximum rank. (k-forms on $\mathbf{R}^2$ vanish if $k \geq 3$).

It is easy to see that it is exact even without using Poincare lemma.

We see that one can find 1-form bdy such that

$$a(x,y)dx \wedge dy = d(b(x,y)dy), \text{ i.e.} \qquad a = \frac{\partial b(x,y)}{\partial x}$$

One can take $a(x,y) = \int_0^x b(t,y)dt$

Hence we see that arbitrary closed 2-form is exact, i.e.

7 *Calculate de Rham cohomology groups for $\mathbf{R}^3$. What can be said about de Rham cohomology groups of the space $\mathbf{R}^3 \backslash 0$ in comparison with $\mathbf{R}^3$ and S^2?*

We know already that $H^0_{DR}(\mathbf{R}^3) = \mathbf{R}$ because $\mathbf{R}^3$ is connected.

$$H^1_{DR}(\mathbf{R}^3) = H^2_{DR}(\mathbf{R}^3) = H^3_{DR}(\mathbf{R}^3) = \mathbf{R} = 0$$

due to Pincare lemma. One can straightforwardly to calculate for closed formes exact formes as in previous exercise.

If we consider $\mathbf{R}^3\backslash 0$ then this space is homotopy equivalent to sphere S^2 and

$$H^0_{DR}(\mathbf{R}^3\backslash 0) = H^0_{DR}(S^2) = \mathbf{R}\,,$$

because sphere and $\mathbf{R}^3\backslash 0$ are connected.

One can show that

$$H^1_{DR}(\mathbf{R}^3\backslash 0) = H^1_{DR}(S^2) = 0 \qquad (*)$$

and

$$H^2_{DR}(\mathbf{R}^3\backslash 0) = H^2_{DR}(S^2) = 0 \qquad (**)$$

Focus attention on the last expression (The previous expression (*) requires additional tools which are out of our programme)

Note that S^2 is *compact oriented manifold*. Hence there exists an volume form on it. In the case of S^2 it is $\sigma = \sin\theta d\theta \wedge d\varphi$:

$$\text{volume of the sphere} = \int \sin\theta d\theta \wedge d\varphi = \frac{4}{3}\pi \neq 0$$

In particularly this means that the form σ is not cohomologous to zero. Indeed suppose $[\sigma] = 0$ in $H^2_{DR}(S^2)$, i.e. $\sigma = dr$. Then due to Stokes Theorem $\int_{S^2} \sigma = \int_{S^2} dr = \int_{\partial S^2} r = 0$. Contradiction. Hence we see that on compact orientable n-manifolds top cohomology group $H^n(M^n)$ is not equal to zero. On the last lecture we considered this question and came to the conclusion that on compact orientable n-manifolds top cohomology group $H^n(M^n) = \mathbf{R}$.

What about $H^3(\mathbf{R}^3\backslash 0)$? Of course it is equal to zero because $H^3(\mathbf{R}^3\backslash 0) = H^3(S^2)$ and for two dimensional manifolds k-dimensional groups vanish for $k \geq 3$.

But one can straightforwardly to perform calculations in spherical coordinates in a way similar to the exercise 3): Let $\omega = a(r,\theta,\varphi)dr \wedge d\theta \wedge d\varphi$ be abn arbitrary 3 form in $\mathbf{R}^3\backslash 0$, where $0 < r < \infty$. To show that this form is exact one have to solve equation:

$$a(r,\theta,\varphi)dr \wedge d\theta \wedge d\varphi = d(b(r,\theta,\varphi)d\theta \wedge d\varphi)$$

Evidently this equation has solution: $b = \int_1^r a(t,\theta,\varphi)dt$.