First Lecture

6/10/2016

States and Observables

Unitary space

Complex vector space $\mathcal{H}$ equipped with scaler product $\langle\ ,\ \rangle$ [$\langle\ ,\ \rangle$ — hermitian positive definite form]:

$$\langle \vec{x}, \vec{y} \rangle = \overline{\langle y, x \rangle}$$

$$\langle \lambda x, y \rangle = \lambda \langle x, y \rangle \qquad (\langle x, \lambda y \rangle = \bar{\lambda} \langle x, y \rangle)$$

$$\langle x, x \rangle \geq 0, \quad \langle x, x \rangle = 0 \Leftrightarrow \vec{x} = 0 .$$

$$|x| = \sqrt{\langle x, x \rangle}$$

Example 1. $\mathbb{C}^2 = \{ \begin{pmatrix} a \\ b \end{pmatrix}, \ a, b \in \mathbb{C} \}$ [$\mathbb{C}^n: \langle x, y \rangle = \sum_{i=1}^{n} x^i \bar{y}^i$]

$$\begin{pmatrix} a \\ b \end{pmatrix} = a \uparrow + b \downarrow .$$

$\{e_i\}$ - orthogonal basis in $\mathcal{H}$: $\quad \langle e_i, e_j \rangle = \delta_{ij}$

Example 2. $V = \{ f : f \in C(\mathbb{R}^3), \ \int f(x) \bar{f}(x) d^3x < \infty \}$

(Square integrable functions)

pre - Hilbert space.

V - is (non-complete) unitary space.

$\bar{V} = L^2(\mathbb{R}^3)$ — square integrable measurable funct.

Hilbert space $\quad |\langle f, g \rangle| \leq |f| \cdot |g|$

Th. Cauchy - Bunyakovsky - Schwarz ineq.

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle$$

(i.e. $\langle\ ,\ \rangle$ defines norm $|\vec{x}| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$ which obeys triangle inequality. $|\vec{x}_1 + \vec{x}_2| \leq |x_1| + |x_2|$.

$$P_\varphi(t) = \langle t x + y e^{i\varphi}, t x + y e^{i\varphi} \rangle$$

$$P_\varphi(t) = t^2 \langle x, x \rangle + t[e^{i\varphi} \langle y, x \rangle + c.c.] + \langle y, y \rangle \geq 0$$

$$\Downarrow$$

C.B.Sch.

We will try to avoid as much as possible discussion of $\dim \mathcal{H} = \infty$, but sometimes...
[$\dim \mathcal{H} < \infty$ — internal degrees of freedom]

Space of States in QM

$\mathcal{H}$ – Unitary space

- State (pure state) = Ray in Hilbert space $\mathcal{H}$ = one-dimensional complex subspace. $= [\Psi]$, $\Psi \in \mathcal{H}$, vector Ψ ($\Psi \neq 0$) defines ray $[\Psi]$

- Superposition of states
  $\Psi = \sum c_i \Psi_i \qquad c_i \in \mathbb{C}$
  Ψ is superposition of states $\{\Psi_i\}$

- If state — $[\Psi]$
  Probability that it will be in state $[\varphi]$ =

  $P_{\psi\ldots}$?!! $\dfrac{|\langle \Psi, \varphi \rangle|}{|\Psi| \cdot |\varphi|}$ = [System Ψ can be obtained in a state φ]

  $\mathbb{C} \ni \langle \Psi, \varphi \rangle$ – probability amplitude.

example

Example

$\mathcal{H} = \mathbb{C}^2 \quad \Psi = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \varphi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \varphi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\langle \Psi, \varphi \rangle = \frac{1}{2}$

Ψ ~~[...]~~

Prepared state Ψ → $\varphi_1 \quad \langle \Psi, \varphi_1 \rangle$

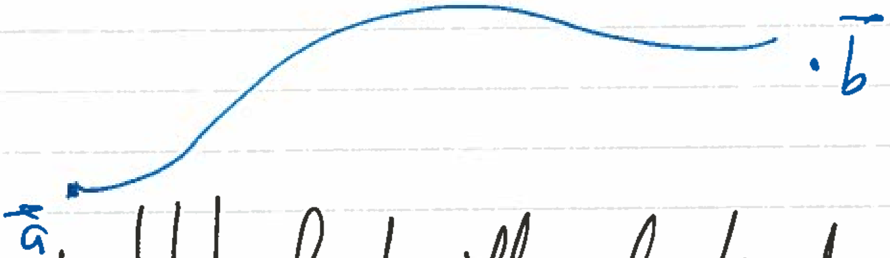
→ $\varphi_2 \quad \langle \Psi, \varphi_2 \rangle$

→ $\varphi_3 \quad \langle \Psi, \varphi_3 \rangle$

Probability amplitude – Feynman Integral

$\mathcal{H} \ni \Psi_{\vec a}$ $\left\{ \begin{array}{l} \text{state character. particle which is} \\ \text{located at the point } \vec a \end{array} \right\}$

$$\langle \Psi_{\vec a}, \Psi_{\vec b} \rangle \sim \int e^{i \int L(x,v)\,dt} \,\mathcal{D}\vec{x}(t)$$



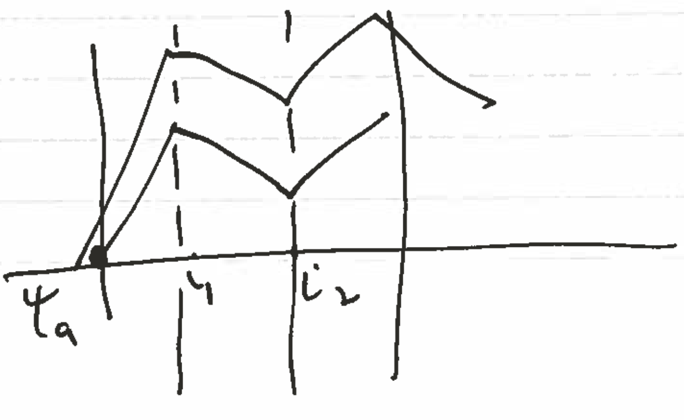
Important !!! but ill-defined.

$\Downarrow$

$$\langle \Psi, \varphi \rangle = \prod_i \langle \Psi, e_i \rangle \langle e_i, \varphi \rangle =$$

$$= \prod_{i,j} \langle \Psi, e_i \rangle \langle e_i, e_k \rangle \langle e_k, \varphi \rangle = \ldots =$$

$$= \prod_{i_1 \ldots i_N} \underbrace{\langle \psi, e_{i_1} \rangle \langle e_{i_1}, e_{i_2} \rangle \ldots \ldots \langle e_{i_{N-1}}, e_{i_N} \rangle \langle e_{i_N}, \psi \rangle}_{\text{product over trajectories}}$$



II

Problem of measurement.

Lattice of Questions.

Classical Mechanics.

Questions {Subsets} — States points

point $\in$ Subset Yes
point $\notin$ Subset No

Q M

Question. ~~Projector~~ Subspace in $\mathcal{H}$. — State ~~Ray~~ spanned by ψ

$\psi \in L \subset \mathcal{H}$ Yes

$\psi \perp L \subset \mathcal{H}$ No

. . .

Princip.

F - physical magnitude $\longrightarrow$ $\hat{F}$ - self-adjoint operator on $\mathcal{H}$

Ψ is a state with magnitude F equal to λ_F if

$\hat{F}\Psi = \lambda_F \Psi$. Ψ is 'eigen vector'

If $\Psi = C_1 \Psi_1 + C_2 \Psi_2$ $\left(\begin{matrix} \hat{F}\Psi_1 = \lambda_1 \Psi_1 \\ \hat{F}\Psi_2 = \lambda_2 \Psi_2 \end{matrix} \right)$

Measuring F in N experiments we will come to

answer $F = \lambda_1$ in n_1 experiments $(|\Psi_1| = |\Psi_2|)$

$F = \lambda_2$ in n_2 experiments.

$n_1 : n_2 = |C_1|^2 : |C_2|^2$ $n_1 + n_2 = N$