III - lecture

III - 0

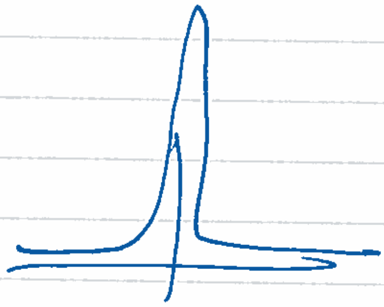
Generalised functions (distributions)

Physical intuition:

density for 'point' particle

$$\rho_a(x) = \frac{1}{a\sqrt{\pi}} e^{-\frac{x^2}{a^2}}$$

$a \to 0$ $\rho_a(x) \to$



$$L_a(\varphi) = \int \rho_a(x)\,\varphi(x)\,dx$$

$$\lim_{a\to 0} L_a(\varphi) = \varphi(0)$$

$$\rho_a(x) \xrightarrow[a\to 0]{} \cancel{???}\ ???$$

???

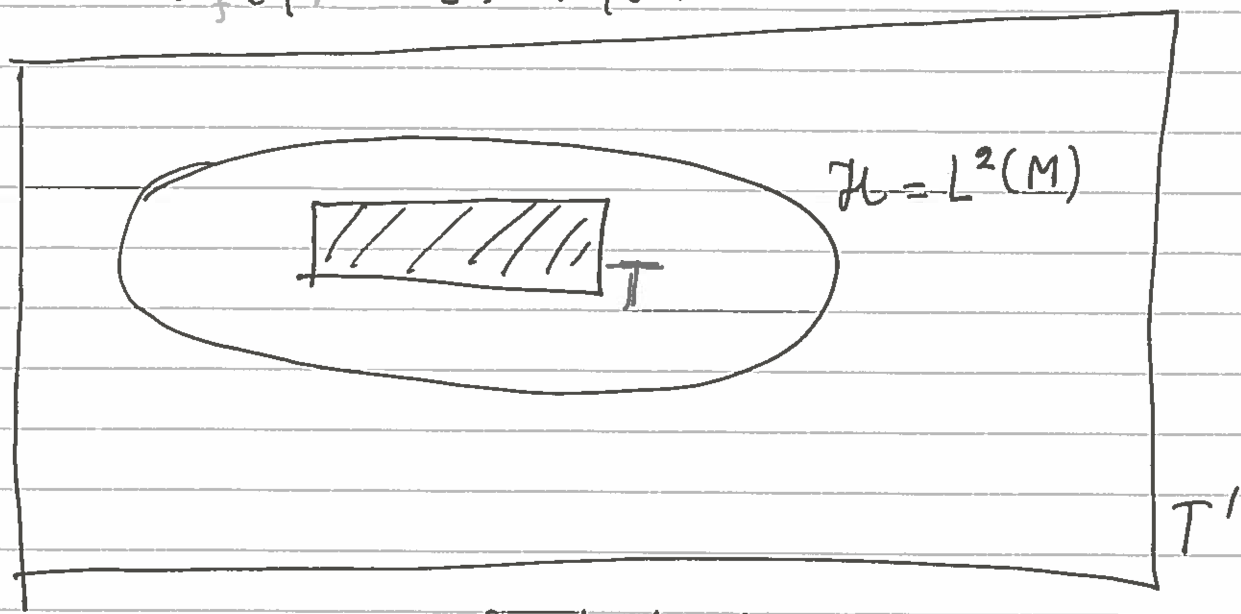
We want

1) $$\lim_{a\to 0} \int \rho_a(x)\,\varphi(x)\,dx = \int \lim_{a\to 0} \rho_a(x)\,\varphi(x)\,dx$$

2) $$\int ???'\,\varphi(x)\,dx = \int ???\,\varphi'(x)\,dx$$

Generalised function (Distributions)

$f \mapsto L_f(\varphi) = \int f(x)\,\varphi(x)\,dx.$



T – subspace of test function

$\underbrace{\text{T' – space of linear functionals on T}}_{\text{generalis. fun-ons, distributions.}}$

$\mathcal{H} = L^2(\mathbb{R})$ $\quad T = \{\text{funct. with compact support}\}$

T' — lin. functionals on T

$\delta(x-x_0)$: $\quad L_f(\varphi) = \int \delta(x-x_0)\,\varphi(x)\,dx$

$f = x^n \quad \int x^n \varphi\, dx.$

$\mathcal{H} = L^2(\mathbb{R}^n)$, $T = \mathcal{D}(\mathbb{R}^n) = \{\varphi : |\sup x^\alpha D^\beta \varphi| < \infty\}$

T' – distribution.

Usually

$T = \{\varphi : |\sup x^\alpha D^\beta \varphi| < \infty\}$

(rapidly decreasing f-ons)

T' – tempered distributions

$f = \delta(\vec{r} - \vec{r}_0)$

$\theta = \begin{cases} 1 & |x| \cancel{<} a \\ 0 & |x| > a. \end{cases}$

$\theta' = \delta(x-a) + \delta(x+a)$

$$\int \delta'(x)\varphi(x) = -\varphi'(x)\big|_{x=0}$$

$$T \subset \mathcal{H}^2(\mathbb{R}^n) \subset T'$$

$T = D(\mathbb{R}^n)$ $\qquad$ $T' = \{\text{gener. function}\}$

$$\boxed{\text{Space of gener. function is closed under taking Fourier transform}}$$

$L_f(\tilde{\varphi}) = L_{\tilde{f}}(\varphi)$ $\left[\begin{matrix} f \to \tilde{f} \\ \varphi \to \tilde{\varphi} \end{matrix}\right]$ Fourier transform

Exercise

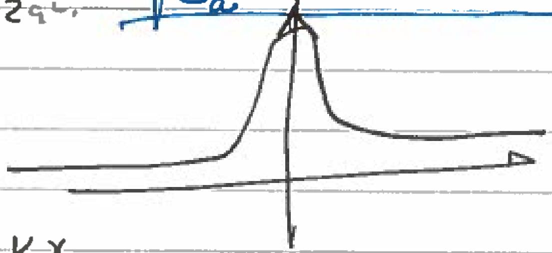
$\int \psi(x)dx = 1$, $C_a =$

$\cancel{\langle \psi_1 \psi \rangle = |C_a|^2 a\sqrt{\pi}}$.

$$\boxed{C_a\, a\sqrt{2}\sqrt{\pi} = 1.}$$

$$\Psi(x) = C_a e^{-x^2/2a^2}$$

$$\psi = \frac{1}{a^{1/2}\pi^{1/4}} e^{-\frac{x^2}{2a^2}}$$



$$\hat{\psi}(k) = \frac{C_a}{\sqrt{2\pi}}\int e^{-\frac{x^2}{2a^2}+ikx}\,dx =$$

$$= \frac{1}{\sqrt{2\pi}} C_a \int e^{-\frac{1}{2a^2}[x+ia^2k]^2 - \frac{a^2k^2}{2}}\,dk =$$

$$= \frac{1}{\sqrt{2\pi}} \underbrace{\boxed{C_a\sqrt{2}\,a\sqrt{\pi}}}_{1}\, e^{-\frac{a^2k^2}{2}}$$

Heisenberg Uncertainty principle

$a \to 0$ $\qquad \longrightarrow \qquad$ $\hat{\varphi}(k) \to 1.$

$\psi(x) \to \delta(x)$

$$\boxed{\delta(x) \longrightarrow 1}$$

Generalised eigen vectors

Let $\mathcal{H}$ - be Hilbert space, (e.g. $\mathcal{H} = L^2(\mathbb{R}^3) = \overline{C(\mathbb{R}^3)}$)
$[\mathcal{H} = \{\Psi(\vec{r}) : \int \bar{\Psi}\Psi < \infty\}]$

M - space (of parameters)

$$\underset{\text{test functions}}{R} \subset L^2(M) \subset \underbrace{R'}_{\text{general. functions.}}$$

We say that $f = f_a(x)$ $(a \in M)$
is generalised eigen vector ~~if~~ of operator A
if
f_a is generalised eigen function on M of with values in Hilbert space $\mathcal{H}$
$\hat{A} f_a = \lambda(a) f_a$
[i.e. $f = f_a(x)$ if $\mathcal{H} = \{\Psi(\vec{r}) \ldots$
such that
for every test function $\varphi(a) \in R$

$$\underbrace{\hat{A} \int f_a \varphi(a) \, d\mu_M}_{\text{vector in } \mathcal{H}} = \underbrace{\int \lambda(a) f_a \varphi(a) \, d\mu_M}_{\text{Vector in } \mathcal{H}},$$

Example $\mathcal{H} = L^2(\mathbb{R}^1) = \overline{C(\mathbb{R}^1)}$
$M = \mathbb{R}^1$
$\hat{A} = \hat{x}$
$f = \delta(x-a)$ is general. eigenf. of $\hat{x}$
$\hat{x}\delta(x-a) = a\,\delta(x-a)$

$$\hat{x} \int \delta(x-a) \varphi(a) \, da = \int a\,\delta(x-a) \varphi(a) \, da$$

$$\| \qquad\qquad \|$$

$$\hat{x} \varphi(x) \qquad\qquad x\varphi(x)$$

Theorem (or unitary)

Let $\hat{F}$ be self-adjoint operator in Hilbert space $\mathcal{H}$. [$\mathcal{H}$ in general is infinite-dimen.]

Then there exists a space M with measure $d\mu_M$ a

$$\mathcal{H} \xleftrightarrow[\text{isomorphism}]{\alpha} \mathcal{H}_M = L^2(M, d\mu_M)$$

such that A becomes operator of multiplication on function $a(x)$

$$\alpha(\hat{A}\Psi) = a(x)\,\cancel{\alpha}\,\alpha(\Psi)$$

[$a(x)$ = real if $\hat{A}^\dagger = \hat{A}$, $|a| = 1$. If A is unitary]

In the case $\dim \mathcal{H} < \infty$,

$$\hat{F}\vec{e}_i = f_i \vec{e}_i$$

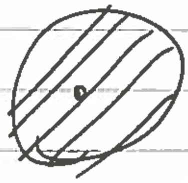
· $f(i)$ — is a function $a(x)$

Exercise

~~Example~~

$$P: \Psi(\vec r) \longrightarrow (P\Psi)(\vec r) = \begin{cases} \Psi(\vec r) & \text{if } \vec r \in B_\varepsilon(\vec r_0) \\ 0 & \text{if } \vec r \notin B_\varepsilon(\vec r_0) \end{cases}$$

P — Question: Does particle belongs to the ball $B_\varepsilon(\vec r_0)$



$\cancel{P\Psi = \Psi}$

$$P\Psi = \Psi \Longrightarrow \text{YES}$$

$$P\Psi \equiv 0 \Longrightarrow \text{No.}$$

For arbitrary Ψ after measurement

Ψ is reduced to $\Psi' = \begin{cases} P\Psi & \text{if Yes} \\ \Psi - P\Psi & \text{if No} \end{cases}$

Momentum

- $\hat{p}_i = \frac{\hbar}{i}\frac{\partial}{\partial x^i}$

$e^{\frac{i\vec{p}\vec{x}}{\hbar}}$ – generalised eigenfunction.

$$\hat{p}_x \int e^{\frac{i\vec{p}\vec{x}}{\hbar}} \varphi(\vec{p})\, d^3p = \int p_x e^{\frac{i\vec{p}\vec{x}}{\hbar}} \varphi(\vec{p})\, d^3p$$

- $$\bar{p}_i = \frac{\langle \Psi, \hat{p}_i \Psi \rangle}{\langle \Psi, \Psi \rangle} =$$

$$= \frac{\int \bar{\Psi}(x) \frac{\hbar}{i}\frac{\partial}{\partial x^i} \Psi(x)\, d^3x}{\int \bar{\Psi}(x) \Psi(x)\, d^3x}$$

Exercise: $\Psi(x)$ is real wave function $\bar{p}$ – ?

Solution.

- $\bar{p}_i = \overline{\frac{\hbar}{i}\int \frac{d}{dx}\Psi\, \Psi dx} = \frac{\hbar}{i}\int \Psi_x \Psi \Rightarrow \bar{p}_i = 0.$

real number

[In fact $\int \Psi_x \Psi = \frac{1}{2}\int (\Psi^2)_x dx = 0$ since $\int \Psi^2 dx < \infty$]

~~Physical~~ . Explanation.

$$e^{\frac{i\vec{p}\vec{x}}{\hbar}} + e^{-\frac{i\vec{p}\vec{x}}{\hbar}} = 2\cos\frac{\vec{p}\vec{x}}{\hbar}$$

] If Ψ is real ~~cont~~ then momenta

$\vec{p}, -\vec{p}$

are contributed

It is phase (gradient of phase) of wave function which contributes to momentum.

$$\Psi(x) = \rho(x)\, e^{i\sigma(x)} \qquad \left[\langle\Psi,\Psi\rangle = \int \rho^2(x)\,dx = 1\right]$$

$$\bar{p} = \langle \Psi, \hat{p}\Psi\rangle = \int \rho(x) e^{-i\sigma(x)} \left(\frac{\hbar}{i}\frac{d}{dx}\right)\left[\rho(x) e^{i\sigma(x)}\right]_x =$$

$$= \frac{\hbar}{i}\Big[\underbrace{\int \rho \rho_x\, dx}_{=0} + \int \rho^2(x)\, i\sigma_x\, dx\Big] = \hbar \int \rho^2(x)\sigma_x\, dx$$

In Quasiclassics

$$\sigma(x) = \frac{S(x)}{\hbar}$$

$$p = \frac{\partial S}{\partial x} \quad \text{[Hamilton-Jacobi]}$$

$$\psi(x) = C e^{-\frac{(x-x_0)^2}{a^2} + \frac{i p_0 x}{\hbar}}$$

$$\overline{X} = X_0 \qquad \overline{X^2} = X_0^2 + \frac{a^2}{2}$$

$$\overline{p} = p_0 \qquad \overline{p^2} = p_0^2 + \frac{\hbar^2}{2a^2}$$

$$\Delta X^2 \cdot \Delta p^2 = \frac{\hbar^2}{4}$$

$a \to 0$

(Heisenberg uncertainty principle)

$(\Delta X^2 \Delta p^2 \geq \frac{\hbar^2}{4})$