IV

Momentum.

$$\hat{p}_i = \frac{\hbar}{i}\frac{\partial}{\partial x^i} \quad \text{on } \mathcal{H} = L^2(\mathbb{R}^3)$$

self-adjoint operator.

$\hat{A}$ self-adjoint on $\mathcal{H}$

$\Downarrow$

$\mathcal{H} = L^2(M)$

$(\hat{A} f)(m) = a(m) f(m)$

According spectral theorem there exist representation such that $\hat{p}_i$ becomes ~~mat~~ operator of multiplication on function.

$\Downarrow$

Fourier transform

$f(x) \in L^2(\mathbb{R})$

$$\hat{f}(k) \quad (\mathcal{F} f)(k) = \frac{1}{\sqrt{2\pi}} \int e^{-ikx} f(x)\, dx$$

$$(\mathcal{F}^{-1} \hat{f})(x) = \frac{1}{\sqrt{2\pi}} \int e^{ikx} \tilde{f}(k)\, dk.$$

$$\langle \mathcal{F} f, g \rangle = \langle f, \mathcal{F} g \rangle$$

$$\int \overline{\mathcal{F} f}\, g = \int \bar{f}\, \mathcal{F} g$$

$\mathcal{F}$ transforms tempered distributions onto tempered distributions.

To see a world in a Grain.

$$(\mathcal{F} 1) = \sim \delta \qquad \mathcal{F} x = \sim \delta'$$

Grain of sand contains Universe and Universe —— grain of sand

To see a World in a Gram of Sand
And a Heaven in a Wild Flower
Hold Infinity in the Palm of your hand
And Eternity in an hour

. William Blake

Let $\hat{A}, \hat{B}$ be two ~~operators~~ observables.

$$[\hat{A}, \hat{B}] = 0$$

$$\Downarrow$$

there exist a basis $\{\varphi_i\}$:

$$\hat{A}\varphi_i = a_i \varphi_i$$

$$\hat{B}\varphi_i = b_i \varphi_i$$

$$\mathcal{H}_i = \{\varphi : \hat{A}\varphi = a_i \varphi\}$$

$$B|_{\mathcal{H}_i} : \mathcal{H}_i \to \mathcal{H}_i$$

$$\hat{A}(\hat{B}\varphi) = a_i \hat{B}\varphi$$

$$[\hat{A}, \hat{B}] \neq 0,$$

$$[\hat{A}, \hat{B}] = i\hat{C} \neq 0.$$

Heisenberg uncertainty principle

Let Ψ: $\quad \langle \Psi, A\Psi \rangle = 0$

$\langle \Psi, B\Psi \rangle = 0$

i.e. $\bar{A} = \bar{B} = 0$

$$\bar{A}^2 \, \bar{B}^2 \geq C$$

$$\bar{A}^2 \, \bar{B}^2 = \langle \Psi, A^2\Psi \rangle \langle \Psi, B^2\Psi \rangle \geq$$

$$\geq |A\Psi|^2 \, |B\Psi|^2 \geq |\langle A\Psi, B\Psi \rangle|^2 \geq$$

$$\geq |\operatorname{Im} \langle A\Psi, B\Psi \rangle|^2 = \frac{1}{4} |\langle \Psi, i[AB]\Psi \rangle|^2 =$$

$$\operatorname{Re} \langle A\Psi, B\Psi \rangle =$$

$$\cancel{\langle A\Psi} \; \langle A\Psi, B\Psi \rangle = \langle \Psi, AB\Psi \rangle = \underbrace{\langle \Psi, \tfrac{1}{2}(AB + BA)\Psi \rangle}_{\text{real}} + \underbrace{\langle \Psi, \tfrac{1}{2}[A,B]\Psi}_{\text{Im.}}$$

$$\boxed{\Delta A^2 \cdot \Delta B^2 \geq \frac{1}{4} \bar{C}^2}$$

Heisenberg uncertainty principle

$$\overline{A} = 0. \qquad (\hat{A} \longrightarrow \hat{A} - \overline{A})$$

$$\overline{B} = 0.$$

$$\overline{A^2} \cdot \overline{B^2} = \langle \Psi, A^2 \Psi \rangle \langle \Psi, B^2 \Psi \rangle =$$

$$= |A\Psi|^2 \, |B\Psi|^2 \geq \left| \langle A\Psi, B\Psi \rangle \right| \geq$$

$$\langle A\Psi, B\Psi \rangle = \mathrm{Re}\langle A\Psi, B\Psi \rangle + i\,\mathrm{Im}\langle A\Psi, B\Psi \rangle$$

$$\geq \left| \mathrm{Im}\langle A\Psi, B\Psi \rangle \right| = \left| \frac{\langle A\Psi, B\Psi \rangle - \langle B\Psi, A\Psi \rangle}{2i} \right| =$$

$$= \left| \frac{\langle \Psi, [A,B]\Psi \rangle}{2i} \right| = \frac{1}{2} \langle \Psi, C\Psi \rangle$$

$$\boxed{\Delta A^2 \, \Delta B^2 \geq \frac{\Delta C^2}{2}}$$