V – 1

Shrodinger equation.

$H = H(p,q)$ – Hamiltonian in classical mechanics.

$[\Psi]$ – state.

evolution in the Unitary operator:

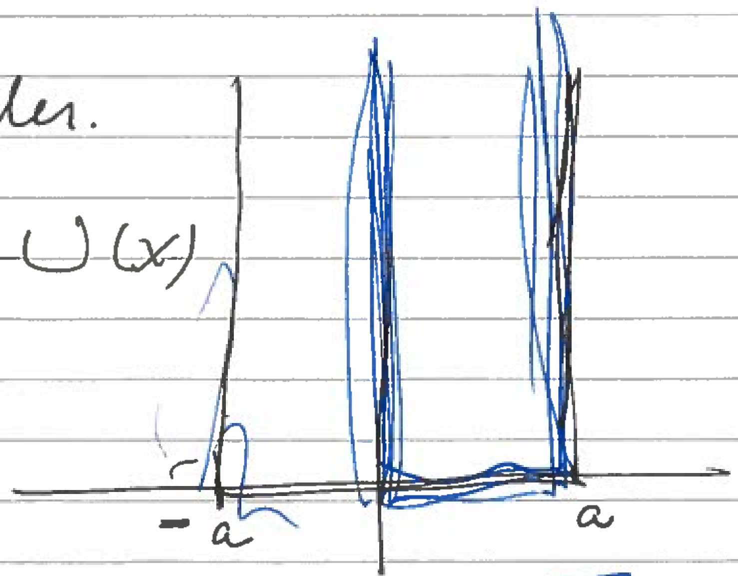
$$\Psi(t) = e^{\frac{1}{i\hbar}\hat{H}t}\,\Psi$$

$$\boxed{i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi}$$

Stationary state Ψ: $\hat{H}\Psi = \lambda\Psi$

Examples.

$$\hat{H} = \frac{\hat{p}^2}{2m} + U(x)$$

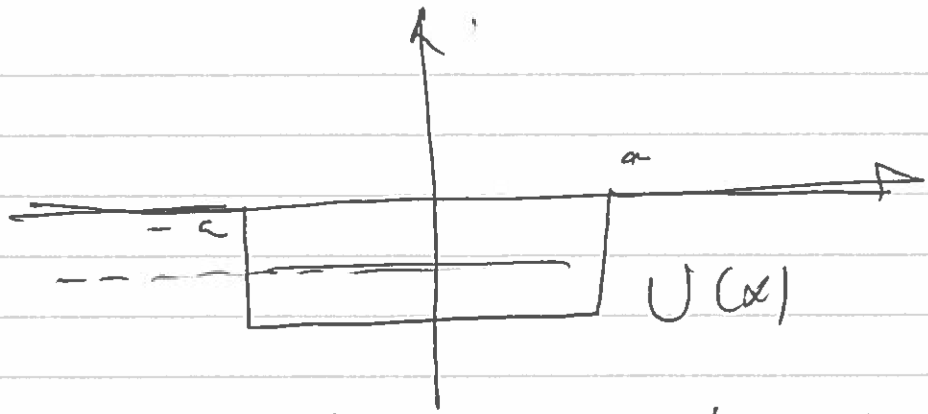


$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} = E_n\Psi$$

$\Psi_n =$

$$\Psi_n = \sqrt{\frac{2}{a}}\,\sin\frac{\pi(n+1)x}{a}$$

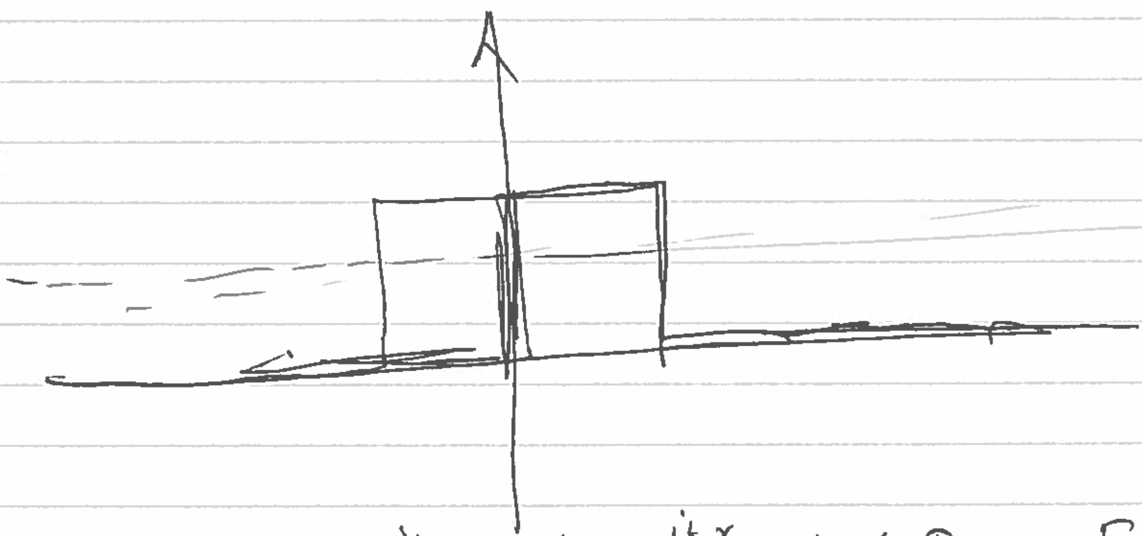
$$E_n = \frac{\pi^2\hbar^2}{2ma^2}(n+1)^2$$



$$-\frac{\hbar^2}{2m}\Psi_{xx} + U(x)\Psi = E\Psi$$

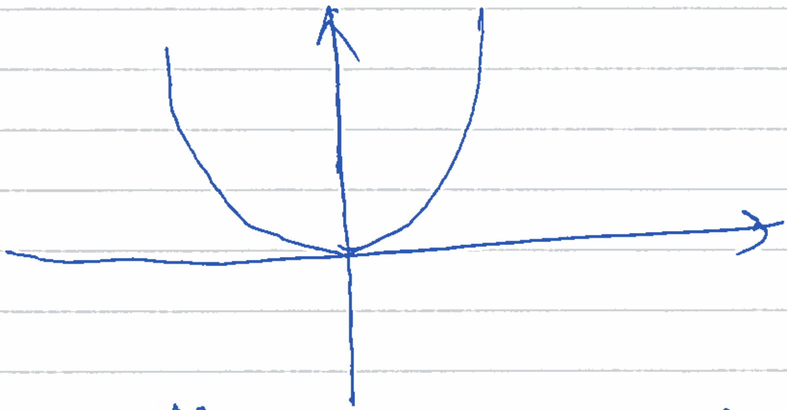
$$\Psi_{xx} = -\frac{2m}{\hbar^2}(E-U)\Psi$$

$$\Psi = \begin{cases} A\cos\sqrt{\frac{2m}{\hbar^2}(U_0-E)}\;x & |x|<a \\ B\exp\left[-\sqrt{\frac{2m}{\hbar^2}E}\;x\right. & \end{cases}$$



$$\Psi = \begin{cases} e^{ikx}+Ae^{-ikx} & x<0 \\ Be^{\varkappa x}+Ce^{-\varkappa x} & 0<x<a \\ Ge^{ik(x-a)} & x>a \end{cases} \qquad E<U$$

Harmonic oscyllator



$$\hat H = \frac{\hat p^2}{2m} + \frac{m\omega^2 q^2}{2} \qquad \hat H \Psi_n = E_n \Psi_n$$

$$q = \sqrt{\frac{\hbar}{m\omega}}\, x, \qquad \frac{d}{dx} = \sqrt{\frac{\hbar}{m\omega}}\frac{d}{dq}$$

$$\cancel{-\frac{\hbar^2}{2m}\frac{\hbar}{m\omega}}$$

$$\left(-\frac{\hbar^2}{2m}\frac{m\omega}{\hbar}\frac{d^2}{dx^2} + \frac{m\omega^2}{2}\frac{\hbar}{m\omega}x^2\right)\Psi = E\Psi$$

$$\frac{1}{2}\left(x^2 - \frac{d^2}{dx^2}\right)\Psi = \frac{E}{\hbar\omega}\Psi$$

ε

$$\boxed{x = \sqrt{\frac{m\omega}{\hbar}}\, q} \qquad \boxed{\varepsilon = \frac{E}{\hbar\omega}}$$

$$\frac{1}{2}\left(x^2 - \frac{d^2}{dx^2}\right)\Psi = \varepsilon\Psi$$

V-4

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)$$

$$\hat{a}^+ = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)$$

$$\hat{x} = \sqrt{\frac{\hbar}{2}\frac{1}{m\omega}}\left(\hat{a} + \hat{a}^+\right)$$

$$\hat{p} = i\sqrt{\frac{\hbar}{2}m\omega}\left(\hat{a}^+ - a\right)$$

$$[\hat{a}, \hat{a}^+] = \frac{m\omega}{2\hbar}\left\{\frac{i}{m\omega}[\hat{p},\hat{x}] - \frac{i}{m\omega}[\hat{x},\hat{p}]\right\} =$$

$$\frac{m\omega}{2\hbar}\left[\frac{i}{m\omega}\frac{\hbar}{i} + \frac{i}{m\omega}\frac{\hbar}{i}\right] = 1$$

$$[\hat{p},\hat{x}] = \frac{\hbar}{i}$$

$$[\hat{a}, \hat{a}^+] = 1$$

$$\hat{a}^+\hat{a} = \frac{m\omega}{2\hbar}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) =$$

$$= \frac{m\omega}{2\hbar}\left(x^2 + \frac{\hat{p}^2}{m^2\omega^2} - \frac{i}{m\omega}\left(\hat{p}\hat{x} - \hat{x}\hat{p}\right)\right] =$$

$$= \frac{m\omega x^2}{2\hbar} + \frac{\hat{p}^2}{2m\hbar\omega} - \frac{i}{2\hbar}\frac{\hbar}{i} = \frac{1}{\hbar\omega}\hat{H} - \frac{1}{2}$$

$$\hbar\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right) = \hat{H}$$

IV – ~~V~~

$\Phi = |0\rangle$ $\qquad H = \hat{a}^\dagger \hat{a} + \frac{1}{2}$

$|1\rangle = \hat{a}^\dagger |0\rangle \qquad \langle 1, 1\rangle = 1.$

$\hat{a}^\dagger |n\rangle = \sqrt{n+1}\, |n+1\rangle$

$\hat{a} |n\rangle = \sqrt{n}\, |n-1\rangle$

$\left|\hat{a}^\dagger |n\rangle\right|^2 = \langle n | a a^\dagger | n\rangle = \langle n | a^\dagger a + 1 | n\rangle =$

$= \langle n | \hat{H} | n\rangle + \frac{1}{2} = (n + \frac{1}{2}) + \frac{1}{2} + (n+1)$

$|0\rangle = e^{-\frac{x^2}{2}}$

$|1\rangle = a^\dagger |0\rangle = \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right) e^{-\frac{x^2}{2}}$

$|2\rangle = \frac{1}{\sqrt{2}} |a^\dagger \cancel{|} 0\rangle$

~~Harmu~~

V - XVI

$$\Big[\underbrace{\frac{1}{\sqrt{2}}\Big(x-\frac{d}{dx}\Big)}_{a^+}\cdot\underbrace{\frac{1}{\sqrt{2}}\Big(x+\frac{d}{dx}\Big)}_{a}+\frac{1}{2}\Big]\Psi=\mathcal{E}\Psi$$

$$\Big(\hat{a}^+\hat{a}+\frac{1}{2}\Big)\Psi=\mathcal{E}\Psi.$$

$$[a,a^+]=\mathbb{1},$$

$$\mathcal{E}=\Big\langle\Big(a^+a+\frac{1}{2}\Big)\Psi,\Psi\Big\rangle=\frac{1}{2}+|a\Psi|^2$$

$$\mathcal{E}\geq\frac{1}{2}$$

$$\Psi=\Phi: \quad a\Phi=0.$$

$\boxed{\Phi - \text{vacuum.}}$

highest weight vector

$a\Phi=0$

$x\Phi=\Phi_x$

$$\boxed{\Phi=Ce^{-\frac{x^2}{2}}}$$

Let $H\Psi=\mathcal{E}\Psi.$

$$\left.\begin{aligned}[H,a^+]&=a^+\\ [H,a]&=-a\end{aligned}\right|\Longrightarrow$$

$$H(\hat{a}^+\Psi)=(\mathcal{E}+1)\hat{a}^+\Psi$$

$$H(\hat{a}\Psi)=(\mathcal{E}-1)\hat{a}\Psi$$

$\hat{a}^+$ – creation

$\hat{a}$ – annihilation.