Lecture VII 1-st December 2016

Angular momentum.

$$\hat{L}_i = [\vec{r} \times \vec{p}] \qquad \hat{L}_i = \frac{\hbar}{i} \varepsilon_{ikm} x_k \partial_m$$

$$[\hat{L}_i, \hat{L}_k] = i \varepsilon_{ikm} \hat{L}_m \quad (\hbar = 1)$$

Recall

$$\Psi(x) = \int \Phi(p) e^{\frac{ipx}{\hbar}} dp = \sum \Phi_p e^{\frac{ipx}{\hbar}}$$

eigenfunction of translation operator

$$T_{\vec{a}} e^{\frac{i\vec{p}\vec{r}}{\hbar}} = e^{\frac{i\vec{p}\vec{a}}{\hbar}} \cdot e^{\frac{i\vec{p}\vec{r}}{\hbar}}$$

Fourier transform is expansion of module $\mathcal{L}^2(\mathbb{R}^3)$ over irreducible representations of translation group.

$$\Downarrow$$

Consider $SO(3)$

$[SO(3), V]$ $SO(3) \ni g \to \Psi^g(\vec{r}) = \Psi(g^{-1}\vec{r})$

Perform expansion of V over irreducible spaces of $SO(3)$

V — functions on S^2

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$K[x,y,z]$ – polynomials on $\mathbb{R}^3$

$$A_m = \{K[x,y,z] \ni P: \quad P(\lambda r) = \lambda^m P(\bar{r})\}$$

A_m - space of homogeneous polynomials of order m

$[SO(3), A_m]$ A_m is invariant subspace

$$H_m = \{P: \quad P \in A_m \quad \Delta P = 0\}$$

$P|_{S^2} = 0$

$\Downarrow$

$P = 0$

if

$P \in A_m$

$$A_2 = H_2 \oplus r^2 H_0$$

$$T_{ik} = \left(T_{ik} - \frac{1}{3}\delta_{ik}T\right) + \frac{1}{3}\delta_{ik}T$$

Generalise this formula for any m

Consider

$$\langle x^k \cancel{y}^m z^n, \; x^{k'} y^{m'} z^{n'} \rangle = k!\,m!\,n!\,\delta_{kk'}\delta_{mm'}\delta_{nn'}$$

$$\left(\frac{\partial}{\partial x}\right)^+ = x$$

$$\Delta^+ = (x^2 + y^2 + z^2)$$

$$\Delta : A_m \longrightarrow A_{m-2} \qquad H_m = \ker \Delta|_{A_m}$$

$$\ker \Delta|_{A_m} \perp \operatorname{Im} r^2|_{A_{m-2}}$$

$$\boxed{A_{m} = H_m \oplus r^2 A_{m-2}}$$

$$A_m = H_m \oplus r^2 H_{m-2} \oplus r^4 H_{m-4} + r^6 H_{m-6} + \dots$$

expansion over spherical harmonics.

All H_i are invariant subspaces.

Theor. H_i are IRREDUCIBLE subspaces!!

<u>Lemma.</u> If $[SO(3), V]$ V-subspace of functions on sphere, then $\exists\, \tilde{f} \in V$ such that $\tilde{f} \neq 0$, $\tilde{f}$ is $SO(2)$ invariant ($\tilde{f} = \tilde{f}(\theta)$)

Proof. Let $f \in V$: $f \neq 0$ in North pole.

Consider $\tilde{f}(\theta) = \frac{1}{2\pi} \int f(\theta, \varphi)\, d\varphi$.

$\tilde{f}(\theta)$ belongs to V $\tilde{f}|_N \neq 0$.

Space A_m possesses $[\frac{m}{2}] + 1$ invariant subspaces.

On the other hand it possesses $[\frac{m}{2}] + 1$ th $SO(2)$-invariant polynomials:

$$\{ z^m,\ z^{m-2} w\bar{w},\ z^{m-4} (w\bar{w})^2, \dots \dots \}$$

$$w = x + iy, \quad \bar{w} = x - iy$$

Hence all H_m are irreducible!

$$\dim H_m = \dim A_m - \dim A_{m-2} =$$
$$= C^2_{m+2} - C^2_m = \underline{2m+1}$$

$\{H_m\}$ – is space of polynomials of $\hat{L} = n$,

$$\hat{L}^2 \Psi = m(m+1)\Psi \quad \Psi \in H_m.$$

$$\hat{L}^2 = -r^2 \Delta + \hat{D}^2 + \hat{D}$$

Legendre polynomials

Every H_m possesses $SO(2)$-invariant polynomial:

$$A_4 = H_4 + r^2 H_2 + r^4 H_0$$

$$H_4 \ni P_4(\theta) = z^4 + a_1 w\bar{w} z^2 + a_2 (w\bar{w})^2$$
$$H_2 \ni P_2(\theta) = z^2 + b w\bar{w}$$
$$H_0 \ni P_0(\theta) = 1$$

These polynomials = Legendre polynomials.
