Quantum mechanics. Problems 1.

1. Let $\mathcal{H}$ be unitary space (compex vector space with Hermitian scalar product) Show that $\langle \mathbf{x}, \lambda\mathbf{y}\rangle = \bar\lambda\langle \mathbf{x}, \mathbf{y}\rangle$

2. Consider the set of the functions

$$\varphi_n(x) = C_n e^{-nx^2}$$

a) find C_n such that $||\varphi_n|| = 1$

b) Show that the sequence $\{\varphi_n\}$ is Cauchy sequence (we suppose that condition a) holds.)

c) Consider a "function" f such that

$$f = \lim_{n\to\infty} \varphi_n$$

Does this function exist?

3. Prove CBS inequality, i.e. for every two vectors

$$|\mathbf{x}|^2 |\mathbf{y}|^2 \geq |\langle \mathbf{x}, \mathbf{y}\rangle|^2 \ ,$$

in the real and in the complex cases

4. Let $\mathcal{H}$ be the space of complex valued functions on $\mathbf{R}$ such for every $\Psi \in \mathcal{H}$,

$$\int_{\mathbf{R}} \Psi(x)\overline{\Psi(x)}dx < \infty\,,$$

Prove that this is a linear space, i.e. for every two functions $\Psi, \Phi \in H$ their linear combination $\lambda\Psi + \mu\Phi \in \mathcal{H}$.

5. Let $\mathcal{H}$ be unitary vector space. Show that the function $d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}||$ defines the metric, i.e. this function is non-negative, summetric ($d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$); $d(\mathbf{x}, \mathbf{x}) = 0$ and

$$d(\mathbf{y}, \mathbf{x}) + d(\mathbf{z}, \mathbf{y}) \geq d(\mathbf{z}, \mathbf{x})\,, (\text{triangle inequality})\,.$$

6. Let $\{\varphi_n\}$ be the orthonormal system in $\mathcal{H}$.

Show that for two arbitrary Ψ, Φ

$$\langle \Psi, \Phi\rangle = \sum_n \langle \Psi, \varphi_n\rangle\langle \varphi_n, \Phi\rangle\,.$$

This imples that

$$\langle \Psi, \Phi\rangle = \sum_n \langle \Psi, \varphi_n\rangle\langle \varphi_n, \Phi\rangle = \langle \Psi, \varphi_n\rangle\langle \varphi_n, \Phi\rangle = \sum_{n_1,n_2} \langle \Psi, \varphi_{n_1}\rangle\langle \varphi_{n_1}, \varphi_{n_2}\rangle\langle \varphi_{n_2}, \Phi\rangle =$$

$$\sum_{n_1,n_2,n_3} \langle \Psi, \varphi_{n_1}\rangle\langle \varphi_{n_1}, \varphi_{n_2}\rangle\langle \varphi_{n_2}, \varphi_{n_3}\rangle\langle \varphi_{n_3}, \Phi\rangle =$$

$$\sum_{n_1,n_2,n_3,n_4} \langle \Psi, \varphi_{n_1}\rangle\langle \varphi_{n_1}, \varphi_{n_2}\rangle\langle \varphi_{n_2}, \varphi_{n_3}\rangle\langle \varphi_{n_3}, \varphi_{n_4}\rangle\langle \varphi_{n_4}, \Phi\rangle = \dots$$