Quantum mechanics. Problems 3.

If $\Psi \in \mathcal{H}$ is a state and A is observable, then the average value $\overline{A} = \left(\overline{A}\right)_\Psi$ of the observable (self-adjoint operator in $\mathcal{H}$)

$$\left(\overline{A}\right)_\Psi = \frac{\langle \Psi, \hat{A}\Psi\rangle}{\langle \Psi, \Psi\rangle}$$

For example if $\mathcal{H}$ is realised as a space of funcions in $\mathbf{E}^3$, i.e. $\Psi = \Psi(x,y,z)$ is a function such that

$$\int_{\mathbf{E}^3} \Psi^*(x,y,z)\Psi(x,y,z)dxdydz < \infty\,. \tag{1}$$

Then the averages of coordinate x is equal to

$$<x>= (\overline{x})_\Psi = \frac{\langle \Psi, x\Psi\rangle}{\langle \Psi, \Psi\rangle} = \frac{\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\,(x\Psi^*(x,y,z)\Psi(x,y,z))}{\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\,(\Psi^*(x,y,z)\Psi(x,y,z))}\,, \tag{2}$$

and the averages of the momentum p_y is equal to

$$<p_y>= \overline{(\hat{p_y})}_\Psi = \frac{\langle \Psi, p_y\Psi\rangle}{\langle \Psi, \Psi\rangle} = \frac{\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \left(\frac{\hbar}{i}\frac{\partial \Psi^*(x,y,z)}{\partial y}\right)\Psi(x,y,z)}{\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\,(\Psi^*(x,y,z)\Psi(x,y,z))}\,. \tag{3}$$

1. Consider the state

$$\Psi = Ce^{-\frac{(\mathbf{r}-\mathbf{r}_0)^2}{2a^2}} = Ce^{-\frac{(x-x_0)^2+(y-y_0)^2(z-z_0)^2}{2a^2}}\,,$$

a) calculate the constant C such that $\langle \Psi, \Psi\rangle = 1$

b) Calculate averages of coordinates x, y, z

c) Calculate averages of momenta p_x, p_y, p_z for this state.

2. Let Ψ be an arbitrary state which is described by the real function in the space $\mathbf{E}^3$:

$$\Psi^* = \Psi$$

Show that averages of momenta vanish for this state.

Explain why the condition (1) is important.

3 Consider $\Psi = e^{-x^2-y^2}z$. Why you cannot you use formula (3) to evaluate the average of the momentum p_z?

4. For the state

$$\Psi(x,y,z) = Ce^{-\frac{(\mathbf{r}-\mathbf{r}_0)^2}{2a^2}+\frac{i\mathbf{p}_0\cdot\mathbf{r}}{\hbar}} = Ce^{-\frac{(x-x_0)^2+(y-y_0)^2(z-z_0)^2}{2a^2}+\frac{ip_{0x}x+ip_{0y}x+ip_{0z}z}{\hbar}}$$

calculate the averages, $\overline{x}$, $\overline{y}$, $\overline{z}$, $\overline{p}_x$, $\overline{p}_y$, $\overline{p}_z$, and the dispersions $\overline{\Delta x^2}$ $\overline{\Delta y^2}$ $\overline{\Delta z^2}$ $\overline{\Delta p_x^2}$ $\overline{\Delta p_y^2}$ $\overline{\Delta p_z^2}$.