Quantum mechanics. Problems 5.

1

`Coordinate and Momentum representation`

$$\Psi(\mathbf{r}) = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}}\int \Phi(\mathbf{p}) e^{\frac{i}{\hbar}\mathbf{pr}} d^3 r\,, \qquad \Phi(\mathbf{p}) = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}}\int \Psi(\mathbf{r}) e^{-\frac{i}{\hbar}\mathbf{pr}} d^3 p\,. \qquad (1)$$

If the vector $\mathbf{x}$ defines a state of quantum mechanical system, then $\mathbf{x} = \Psi(r)$ is the representation of this state in coordinates, and $|\Psi(\mathbf{r}_0)|^2$ says about a probability to find a particle in the vicinity of the point $\mathbf{r}_0$; respectively If $\mathbf{x} = \Phi(p)$ is the representation of this state in momenta, and $|\Phi(\mathbf{p}_0)|^2$ 'says' about a probability that momentum of the particle is the vicinity of the $\mathbf{p_0}$.

a) Check that formulae (1) are well defined, in particularly check that double Fourier transform in equation (1) is the identity operation.

Hint: you may use the formula that $\int_{-\infty}^{\infty} e^{ikx} dk = 2\pi\delta(x)$.

$$\Psi \to \Phi \to \Psi$$

b) Find momentum representation for wave function $\Psi(x) = \delta(x - x_0) + \delta(x - x_1)$. Explain why average momentum of this state vanishes.

c) Find momentum representatin for the wave function $\Psi(x) = \begin{cases} 1 & \text{for } 0 < x < a \\ 0 & \text{for } x \notin (0, a) \end{cases}$

2

a) Calculate

$$e^{\frac{i}{\hbar}\hat{p}_x} e^{-(x-x_0)^2}\,.$$

b) You can see the following exercise in Quantum Mechanics:

`Show that`

$$e^{\frac{i}{\hbar}\hat{p}_x}\Psi(x) = \Psi(x+1)$$

Is this statement correct (`for an arbitrary smooth function`)?

3

a) Let A be an antisymmetric matrix. Show that e^A is an orthogonal matrix.

b) Calculate the mtrix $e^{t\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}$.

c) Let B be hermitian matrix. Show that e^{iB} is unitary matrix.

d) Calculate the matrix $e^{i\varphi\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}$.

4 At the moment $t = 0$, the wave function of the free particle is

$$\Psi(x) = e^{\frac{-(x-x_0)^2}{2a^2}}$$

How look this wave function at the moment $t = t_0$?

5

a) Find stationary states of the particle in the infinite potential well of the width a.

b) Find the momeuntum representation of the state with minimum energy.

c) Calculate the average momentum of the arbitrary stationary state.