Quantum mechanics. Problems 6.

We consider the Hamiltonian of harmonic oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{mw^2\hat{q}^2}{2} = \hbar w \left(\hat{a}^+\hat{a} + \frac{1}{2} \right) ,$$

where $\hat{a}, \hat{a}^+$ *are annihilation and creation operators:*

$$\hat{a} = \sqrt{\frac{mw}{2\hbar}} \left(\hat{q} + \frac{i}{mw}\hat{p} \right) , \hat{a}^+ = \sqrt{\frac{mw}{2\hbar}} \left(\hat{q} - \frac{i}{mw}\hat{p} \right) ,$$

$$[\hat{a}, \hat{a}^+] = 1 , \qquad \hat{a}|0>= 0$$

where $|0>$ *is the* `vacuum` *vector*

1 Show that

$$[\hat{H}, \hat{a}] = -\hat{a} , \quad [\hat{H}, \hat{a}^+] = \hat{a}^+$$

2 Show that the vector $\varphi = \hat{a}|0>$ is non-zero vector, and caclulate its eigenvalue.

3 Let $\Psi \neq 0$ be an eigenvector of operator H with eigenvalue E.

a) Show that the vector $\hat{a}^+\Psi \neq 0$

b) Show that the vector $\hat{a}^+\Psi$ is also the eigenvector of the operator H and calculate its eigenvalue.

c) Show that the vector $\hat{a}\Psi$ is non-zero vector if and only if $\Psi \neq |1>$, and calculate its eignevalue in this case.

4 Write down in the coordinate representation, the expressions for wave functions of of harmonic oscyllator

a) for the vacuum state

b) for the first, second states and third state

5 Write down the expressions for wave functions of harmonic oscillator in momentum representation for first few states.

Compare the answers with answers of exercise (4).

6 Denote by $|n>$ the n-state of the harmonic oscillator, which has unit norm: $<n|n>= 1$.

a) Show that

$$\hat{a}^+|n>= \sqrt{n+1}|n+1> , \quad \hat{a}|n>= \sqrt{n}|n-1> \text{ (if } n \neq 0 .)$$

b) show that

$$|n>= \frac{(a^+)^n}{\sqrt{n!}}|0> \ .$$

c) show that

$$H|n>=\hbar w\left(n+\frac{1}{2}\right)|n>,$$

i.e. the energy of n-th state is equal to $E_n=\hbar w\left(n+\frac{1}{2}\right)$.