Quantum mechanics. Problems 7.

`Secondary Quantisation`

We consider the Hamiltonian

$$\hat{\mathcal{H}} = \sum_i \left(\frac{\hat{p}_i^2}{2m} + \frac{m w_i^2 \hat{q}_i^2}{2}\right) = \sum_i E_i \left(a_i^+ a_i + \frac{1}{2}\right), \quad \sum_i \hbar w_i \left(a_i^+ a_i + \frac{1}{2}\right), \quad E_i = \hbar w_i \tag{1}$$

This Hamiltonian simultaneousy describes the following two pictures

I *free (non-interacting) harmonic osicillators, every oscillator with frequency w_i;*

II *free non-relativistic identical bosonic particles. Each of these particles is described by the Hamiltonian*

$$H_{\text{classical}} = \frac{p^2}{2m} + U(q)\,, \qquad H_{\text{quantum}} = \frac{\hat{p}^2}{2m} + U(\hat{q})\,, \tag{2}$$

In the second picture classical equations of motion Equations of motion are $\begin{cases} \dot{p} = -\frac{\partial H}{\partial q} = -\frac{\partial U}{\partial q} \\ \dot{q} = \frac{\partial H}{\partial q} = p \end{cases}$ *of one particle after qunatisation become the following*

$$i\hbar \frac{\partial \Psi(q,t)}{\partial t} = \hat{H}\Psi\,, \qquad \Psi(q,t) = \sum_i c_i(t)\varphi_i(x)\,, \quad \text{where } \hat{H}\varphi_n = E_n \varphi_n\,. \tag{3}$$

1 Show that $\{c_n(t)\}$ in equation (3) obey the differential equations $\frac{dc_n(t)}{dt} = E_n c_n(t)$.

2 Show that these equations are equations of motion of classical Hamiltonian

$$\mathcal{H} = \mathcal{H}(c, c^*) = \sum_i E_i c_i^* c_i\,, \quad \text{with Poisson bracket} \{c_i, c_j\} = \{c_i^*, c_j^*\} = 0\,, \quad \{c_i, c_j^*\} = \frac{1}{i\hbar}\delta_{ij}\,.$$

3 Show that the Hamiltonian (1) describing quantum osillators will be the quantum Hamiltonian describing free particles.

4 Consider a state Ψ such that

i) the first oscillator is in the first state, $n_1 = 1$, i.e. its energy is equal to

$$\hbar w_1 \left(n_1 + \frac{1}{2}\right) = \frac{3}{2}\hbar w_1\,,$$

ii) the second oscillator is in the second state, $n_2 = 2$, i.e. its energy is equal to

$$\hbar w_1 \left(n_2 + \frac{1}{2}\right) = \frac{5}{2}\hbar w_1\,,$$

all other oscillators are in the ground state: $n_3 = n_4 = \ldots = 0$, i.e. their energies are equal to respectively to $\frac{1}{2}\hbar w_i$.

a) write down the wave function $\Psi = \Psi(x_1, x_2, x_3, \ldots)$ of these osillators in coordinate representation

b) write down the wave function $\hat{a}_2 \Psi$

the wave function $\Psi = |12>$ corresponds in the second picture to the wave function of 3 particles: one particle at the energy E_1 and two particles at the energy E_2

c) write down this wave-function in terms of wave functions $\{\varphi_i(x)\}$ (eigenfunctions of one particle: $\hat{H}\varphi_n = E_n \varphi_n$.)