Quantum mechanics. Problems 8.

`Angular momentum`

Let

$$A_m = \{\text{space of polynomials of order } m\}\,, A_m \ni P = \sum P_{i_1 i_2 \dots i_m} x^{i_1} x^{i_2} \dots x^{i_m},\ i_k = 1, 2, 3\,,$$

and H_m be subspace of harmonic (traceless) polynomials in A_m:

$$H_m = \{P\colon\ P \in A_m\,, \Delta P = 0\}\,,$$

i.e.

$$A_m \ni P = \sum P_{i_1 i_2 \dots i_m} x^{i_1} x^{i_2} \dots x^{i_m}\colon\ \Delta P = \sum_m \frac{\partial P(x,y,z)}{\partial x^m \partial x^m} = P_{m m i_3 i_4 \dots i_k} = 0,.$$

We have that $2m+1$-dimenisonal space H_m is irreducible representation of the group $SO(3)$.

1 In the case if wave function is equal to

a)

$$\Psi = \Psi(\mathbf{r}) = F(r)(ax + by + cz)$$

b)

$$\Psi = \Psi(\mathbf{r}) = F(r)(x^2 + y^2 + \varepsilon z^2)\,, (\varepsilon = \pm 1)$$

$$\Psi = \Psi(\mathbf{r}) = F(r)(x^3 + y^3)\,,$$

where $F(r)$ is the function of $r = \sqrt{x^2 + y^2 + z^2}$ be wave function of the particle.

Calculate angular momentum and the projection of angular momentum on the axis z.

Calculate the action of operator $\hat{L}^2$ on the wave function

2 Show that arbitrary wave function $\Psi(r)$ can be represented in the following way

$$\Psi(\mathbf{r}) = A^{(0)}(r) + A^{(1)}_i(r) x^i + A^{(2)}_{ik}(r) x^i x^k + A^{(3)}_{ikm}(r) x^i x^k x^m + \dots\,,$$

where $A^{(m)}_{i_1 \dots i_m}$ is symmetric traceless tensor.

What is the meaning of this expansion.

3 Perform the expansion described in the previous problem for the function $\Psi(\mathbf{r}) = x^3$. What is the meaning of this expansion

4 a) Calculate explicitly the operator

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$

in Cartesian coordinates.

b) Show that

$$\hat{L}^2 = -r^2\Delta - \hat{E}^2 - \hat{E}\,,$$

where

$$\hat{E} = x\partial_x + y\partial_y + z\partial_z$$

is Euler operator

c) Show that for arbitrary $P \in H_m$

$$\hat{L}^2 P = m(m+1)P\,.$$

d) Show that for arbitrary $P \in H_m$ and arbitrary function $F = F(r)$

$$\hat{L}^2 \Psi P = m(m+1)\Psi\,, \text{ if } \Psi = F(r)P\,.$$