27 September 2018
First lecture

States and Observables

Unitary space.

Complex vector space $\mathcal{H}$ equipped with scalar product $<\ ,\ >$
(hermitian positive definite form)

$$\overline{<X, Y>} = \overline{<Y, X>}$$

(1) $<\lambda X, Y> = \lambda <X, Y>$
(2) $<X, \lambda Y> = \bar{\lambda} < X, Y>$

~~(In physics sometimes is: $<X, \lambda Y> = \lambda <X, Y>$)~~

$\mathcal{E}$
Exercise. Deduce (1) ⟹ (2)

[Remark (In physics ~~case~~ use $<\lambda X, Y> = \bar{\lambda} <X, Y>$

Examle

$$\mathbb{C}^2 = \left\{ \begin{pmatrix} a \\ b \end{pmatrix}, a, b \in \mathbb{C} \right\}$$

$$< \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a' \\ b' \end{pmatrix} > = a\overline{a'} + b\overline{b'}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = a \uparrow + b \downarrow \qquad \left(\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

Sometimes $\mathbb{C}^2$ is called the space of spinors

Example 2

$$V = \left\{ f : f \in C^\infty(\mathbb{R}^3), \text{ such that } \int f \bar{f}(x) dx < \infty \right\}$$

$$< f, g > = \int f(x) \overline{g(x)} d^3 x$$

Exercise Show that ~~V is~~ this is scalar product

Remark The space $(C^\infty(\mathbb{R}^3), \langle , \rangle)$ is not complete. Mathematicians say that this is so called <u>pre-Hilbert space</u>. ②

Hilbert space is a completion of pre-Hilbert space; ~~with respect~~ to i.e. ~~see~~ with respect to metric induced by scaler product.

Recall it:

We say that $\{x_n\}$ is Cauchy sequence if

$$\lim_{n,m\to\infty} \|x_n - x_m\| = 0 \quad \left(\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle}\right)$$

pre-hilbert space $\mathcal{H}$ is Hilbert if it is closed, i.e. every Cauchy sequense has limit

$$\overline{C^\infty(\mathbb{R}^n), \langle \ \rangle} = L^2(\mathbb{R}^n)$$

Physisists usually do not care.

~~Exercise. Let f be a real function. Show that $\langle f, f \rangle$.~~

Exercise

$$\forall f, \int f\bar{f}\,dx < \infty \implies \forall f, g \int f\bar{g} < \infty$$

or in other words

Does scaler product exist?

To answer this question consider

Cauchy – Bunyakovsky – Schwarz inequality

$$\langle \ ,\ \rangle$$

$$\|\langle f, g\rangle\|^2 \leq \|f\|^2 \|g\|^2$$

Proof.

It is instructive first to prove for real case.

$$P(t) = \langle t f + g, t f + g\rangle \geq 0$$

$$\Downarrow$$

CBS.

Complex case

$$P(z) = \langle z f + g, z f + g\rangle =$$

$$= (x^2 + y^2) A + x b - y \beta + C. =$$

$A = \langle f, f\rangle$ $b + i\beta = \langle f, g\rangle$, $C = \langle g, g\rangle$
$z = x + iy$

$$= A\left(x + \frac{b}{A}\right)^2 + A\left(y - \frac{\beta}{A}\right)^2 + C - \frac{b^2}{A} - \frac{\beta^2}{A}.$$

(the one $A = 0 \Rightarrow f = 0$ evident)

$$\Downarrow$$

$$CA > b^2 + \beta^2. \quad \blacksquare$$

(Details on tutorial)

$$\boxed{\begin{array}{c} f, g \in \mathcal{H} \Leftrightarrow \int f \bar{f}\, d\alpha < \infty,\ \int g \bar{g} < \infty \\ \Downarrow \text{CBS} \\ \sqrt{\langle f, g\rangle} < \langle f, f\rangle \langle g, g\rangle < \infty \end{array}}$$

Remark - $\mathcal{H}$ - is called sometimes Hilbert space.

R. We avoid discuss. of difference between finite and inf. dim. case.

Space of states in QM

$(\mathcal{H}, \langle \, , \, \rangle)$ - unitary space (Hilbert space)

State (pure state) - one-dimensional complex subspace $[\Psi]$, $\Psi \in \mathcal{H}$, $\Psi \neq 0$.

Ψ is called Ψ- function.

Superposition of states.

$$\Psi = c_1 \Psi_1 + \ldots + c_n \Psi_n.$$

Postulat: ~~Ψ-fu~~ Space of states is complex linear space (superposition postulate).

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Measurement

In classical mechanics

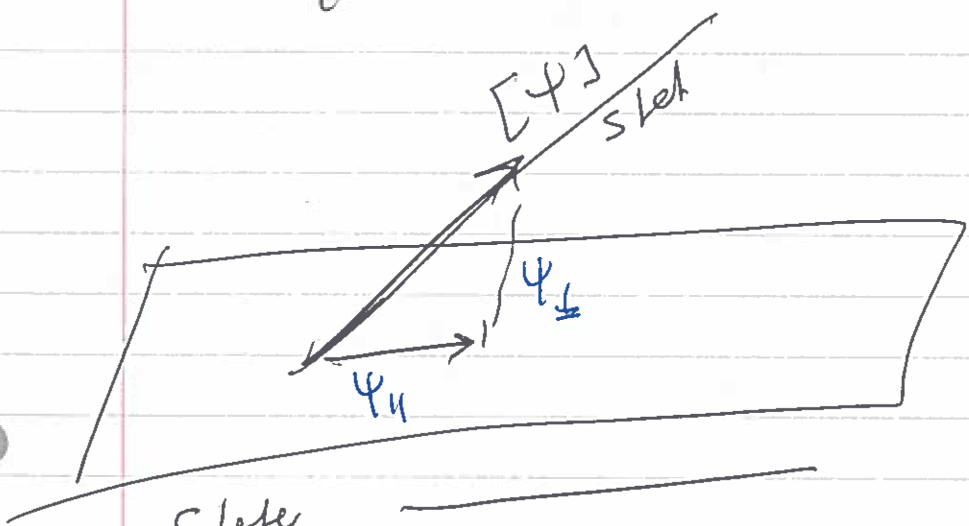
state ——— "point" p

Measurement (Experiment) ——— set A

$p \in A$ Yes

$p \notin A$ No

In Quant. Mec.



State ——— $[\Psi]$

Experiment ——— subspace

Take $\Psi_0 \in [\Psi]$; $|\Psi| = 1$

$$\Psi_0 = a\Psi_{0\parallel} + b\Psi_{0\perp} \qquad (|\Psi_{0\parallel}| = |\Psi_{0\perp}| = 1)$$

$$|a|^2 + |b|^2 = 1.$$

"Probability" that answer is Yes

is equal to $p = |a|^2$.

If we perform $\underline{N}$ experiments with the same data then

in pN experiments answer "Yes"

in $(1-p)N$ " ——————— " "No"

Classical ——— Quantum Mech.

-al Logic

Question

"Set" subspace $V < \mathcal{H}$

State.

a point — Result of experim. — ray in $\mathcal{H}$.

point $\in$ set "yes" $\psi \in V$ "yes"

$\notin$ set 'no' $\psi \perp V$ "no"

$\angle \psi, V \neq 0, \frac{\pi}{2}$ yes or no

Set of question is — Set of question

Distributive lattice — Modular lattice

$A \cap (B \cup C) = A \cap B \cup A \cap C$

$(A \cap (B \cup C) \supset A \cap B \cup A \cap C)$ for arbitrary

$X \leq b \Rightarrow$

$X \vee (A \wedge B) \geq (X \vee A) \wedge b$

Classical Mech. ——— Quantum Mec

Newton equation

$\vec{F} = m\ddot{\vec{q}}$

$F = -\frac{\partial U}{\partial x^i}$

$m\ddot{x}^i = -\frac{\partial U}{\partial x^i}$

Invariant of $SO(3)$

Lagrangian

$L = \frac{m\dot{x}^2}{2} - U(x)$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = \frac{\partial L}{\partial x^i}$ (*)

Invariant of all diffeomorphism.

$$\boxed{\begin{aligned} x &= x(q,t) \\ \dot{x} &= x'(q)\dot{q} + x_t \end{aligned}}$$

$$S(x,t) = \inf_{x(t):\ x(0)=0,\ x(t)=r} \int_0^t L(x(t), \dot{x}(t), t)\, dt$$

$S(x,t)$ — classical action,

[for free particle $S = \frac{m\dot{x}^2}{2t}$]

Invariance of solutions. Eq. are defined by variational principle.

$$S(x,t):\quad \frac{\partial S}{\partial t} + H\left(x, p = \frac{\partial S}{\partial x}\right) = 0$$

$H(x,p)$ - Hamiltonian.

The role of L and H is revealed in Quantum Mechanics.

Variational princip $\Longrightarrow$ Euler Lagrange eq.

$$x \to x + \varepsilon h$$

$$S(x+\varepsilon h) = \int L(x+\varepsilon h, \dot{x}+\varepsilon \dot{h})\,dt =$$

$$= \underbrace{\int L(x,\dot{x})\,dt}_{S[x_0(t)]} + \varepsilon \int \left(\frac{\partial L}{\partial x} h + \frac{\partial L}{\partial \dot{x}} \dot{h} \right) =$$

$$= S[x_0] + \underbrace{\varepsilon \frac{\partial L}{\partial \dot{x}} h \Big|_0^T}_{0} + \varepsilon \int \underbrace{\left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \right)}_{} h$$

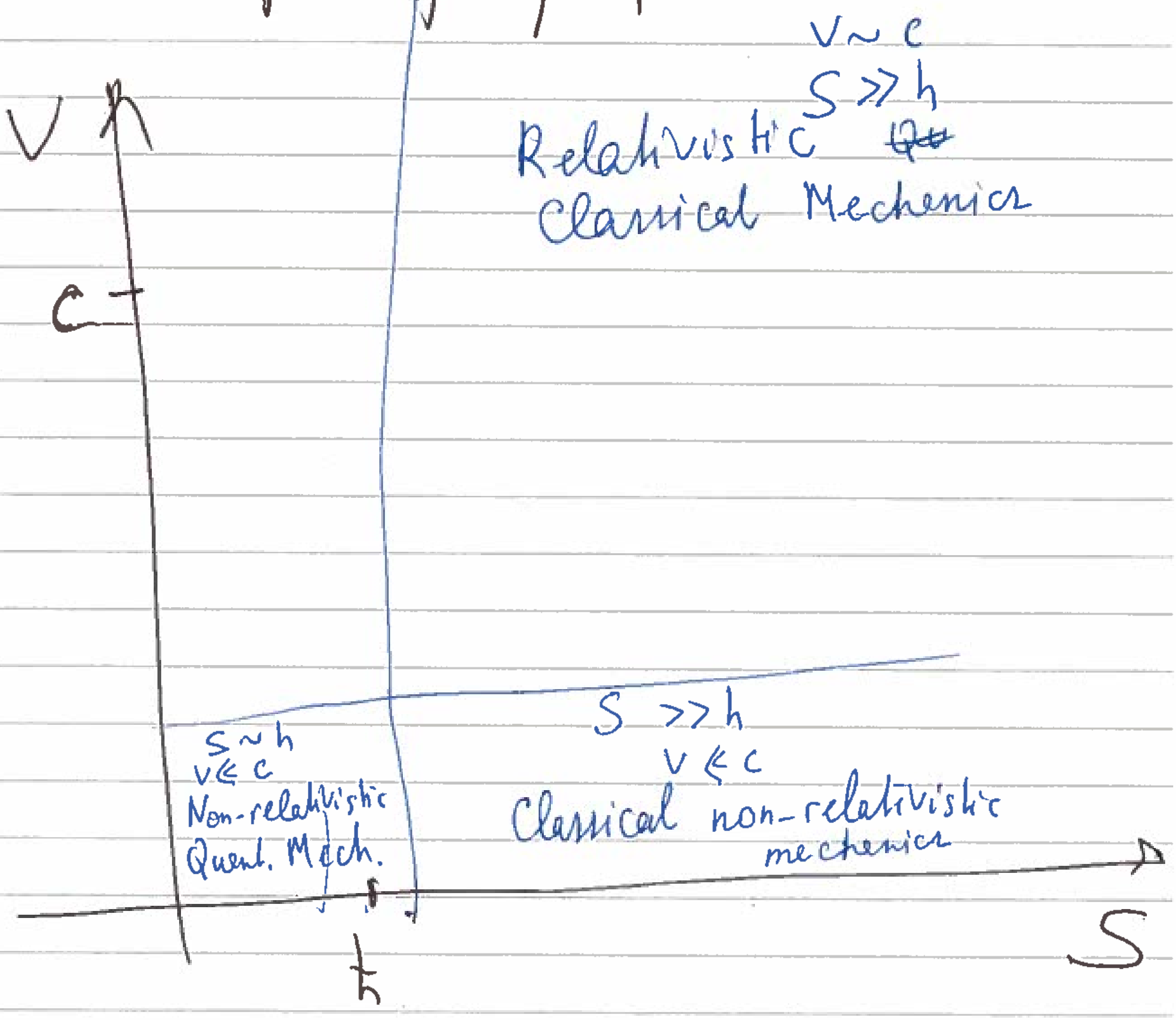
$$x_0 : \quad \frac{\partial L}{\partial x_0} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_0}\right)$$

Euler-Lagrange equations do not depend on coordinates.

$$\Psi \sim e^{\frac{iS}{\hbar}}$$

$$\psi = \int e^{\int L(x,\dot{x})\,dt}\,\mathcal{D}x(t)$$

Map of physics



$\hbar = 6 \cdot 10^{-27}$ эрг/sec

1 эрг $= 10^{-}$ дж.