4 October 2018

Second lecture

$\mathcal{H}$ – Unitary vector space (space of sets)

F – physical magnitude

$\Downarrow$

We assign to F, self-adjoint operator $\hat{F}$ in $\mathcal{H}$

$$\langle \varphi, \hat{F}\psi \rangle = \langle \psi, \hat{F}\varphi \rangle$$

Let system be "prepered" in the state ψ (More precisely $[\psi]$)

We want to measure the magnitude F.

Suppose we perform N experiments

Let $\{\varphi_i\}$ be orthonormal basis of eigenvectors of $\hat{F}$:

$\langle \varphi_i, \varphi_j \rangle = \delta_{ij}$, $\quad \hat{F}\varphi_i = f_i \varphi_i$

$\{\varphi_i\}$ – a basis

If $\psi = \varphi_k$ then value of magnitude F is equal to $\boxed{f_k}$ (in any experiment)

If $\quad \psi = \sum c_i \psi_i = c_1\psi_1 + c_2\psi_2 + \dots$

then F takes values $f_1, f_2, f_3, \dots$

with the probability proportional to

$|c_1|^2, \ |c_2|^2, \ |c_3|^2, \ \dots$

In n_1 experiments the value of F is f_1 2)

In n_2 experiments the value of F is f_2

In n_3 experiments the value of F is f_3

. . . .

In n_K experiments the value of F is f_K

f_K is eigen~~Value~~ of $\hat{F}$ with eigenvector φ_i

$n_1 + n_2 + n_3 + \ldots + n_k + \ldots = N$ (number of experim)

All the experiments are IDENTICAL

$$\cancel{\bar{F} = \frac{\sum n_m f_m}{\sum n_m}} =$$

$n_1 \sim |c_1|^2, \quad n_1 = K|c_1|$

$n_2 \sim |c_2|^2, \quad n_2 = K|c_2|^2$

. . .

$n_K \sim |c_K|^2, \quad n_K = K|C_m|^2$

$$\bar{F} = \frac{\sum n_m f_m}{\sum n_m} = \frac{K\sum|C_m|^2 f_m}{K\sum|C_m|^2} = \frac{\sum f_m|C_m|^2}{|C_m|^2} =$$

$$= \frac{\langle \Psi, \hat{F}\Psi \rangle}{\langle \Psi, \Psi \rangle}$$

Math. Appendix.

$\hat F$ – self-adjoint operator in $\mathcal{H}$.

If $\dim \mathcal{H} < \infty$ then $\exists \{\varphi_i\}$: 1) $\{\varphi_i\}$ – basis

2) $\langle \varphi_i, \varphi_j \rangle = \delta_{ij}$

3) $\hat F \varphi_i = f_i \varphi_i$

Proof

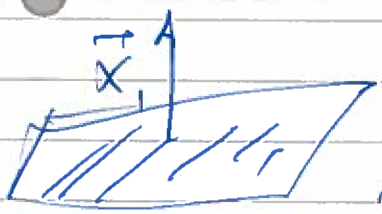
Consider

$$f(\vec\alpha) = \langle \vec X, \hat F \vec X \rangle, \quad |\vec\alpha| = 1$$

$\vec X_1 = \{\vec X : , F(\vec\alpha) - \text{is maximum on } S$

$\Downarrow$

$\vec X_1$ – eigenvector.



$$\lambda_i \neq \lambda_j \Rightarrow \langle \varphi_i, \varphi_j \rangle = 0$$

$$\langle \varphi_i, \varphi_j \rangle = \frac{1}{\lambda_i} \langle \hat F \varphi_i, \varphi_j \rangle = \frac{1}{\lambda_i} \langle \varphi_i, \hat F \varphi_j \rangle = \frac{\lambda_j}{\lambda_i} \langle \varphi_i, \varphi_j \rangle$$

$$\Downarrow$$

$$\varphi_i \perp \varphi_j$$

Let $\{F_1, \dots, F_k\}$ be a set of commuting self-adjoint operators

$\Downarrow$

$\exists$ orthonormal basis $\{\varphi_i\}$,

$$\hat F_{(a)} \varphi_i = \lambda_{(a)i} \varphi_i$$

~~these observables~~ ~~are simultaneously~~ can be measured simultaneously.

[Example – roots of Lie algebra – states such that elements of Cartan ~~alg~~ subalgebra are measurable on them]

Example $\mathcal{H} = \mathbb{C}^2$

$$\Psi = \begin{pmatrix} a \\ b \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix} = a\uparrow + b\downarrow$$

$$\hat{S}_x = \frac{1}{2}\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\sigma_x}, \quad \hat{S}_y = \frac{1}{2}\underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{\sigma_y}, \quad \hat{S}_z = \frac{1}{2}\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\sigma_z}$$

$\hat{S}_x$ – measures x-component of spinor

$\hat{S}_y$ – measures y-component of spinor

$\hat{S}_z$ – measures z-component of spinor

$\sigma_x, \sigma_y, \sigma_z$ – Pauli matrices

$$[S_k, S_m] = \varepsilon_{kmn} S_n$$

(iS_x, iS_y, iS_z – generators of Lie algebra $su(2)$).

Let $\Psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \uparrow$

$$\hat{S}_z\uparrow = \frac{1}{2}\uparrow, \quad \hat{S}_x\uparrow = \frac{1}{2}(\uparrow + \downarrow), \quad \hat{S}_y\uparrow = \frac{1}{2}i(\uparrow + \downarrow)$$

S_z – component of Ψ is equal to $1/2$

S_x – " ———— " is equal to $1/2$ with probability $1/2$, is equal to $-1/2$ with probability $1/2$,

If $\Psi = \sum C_m \Psi_m$ and F the value of F is f_m, then after this measurement system will be in the state $\Psi' = \Psi_m$

$$\Psi = \begin{pmatrix} c_+ \\ c_- \end{pmatrix} = c_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_- \begin{pmatrix} 0 \\ 1 \end{pmatrix} = c_+ \uparrow + c_- \downarrow$$

$$\bar{S}_x = \frac{\langle \Psi, \hat{S}_x \Psi \rangle}{\langle \Psi, \Psi \rangle} = \frac{\left\langle \begin{pmatrix} c_+ \\ c_- \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_+ \\ c_- \end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix} c_+ \\ c_- \end{pmatrix}, \begin{pmatrix} c_+ \\ c_- \end{pmatrix} \right\rangle} =$$

$$= \frac{\left\langle \begin{pmatrix} c_+ \\ c_- \end{pmatrix}, \frac{1}{2}\begin{pmatrix} c_- \\ c_+ \end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix} c_+ \\ c_- \end{pmatrix}, \begin{pmatrix} c_+ \\ c_- \end{pmatrix} \right\rangle} = \frac{1}{2} \frac{\bar{c}_+ c_- + \bar{c}_- c_+}{\bar{c}_+ c_+ + \bar{c}_- c_-}$$

$$\bar{S}_y = \frac{\langle \Psi, \hat{S}_y \Psi \rangle}{\langle \Psi, \Psi \rangle} = \frac{\left\langle \begin{pmatrix} c_+ \\ c_- \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} c_+ \\ c_- \end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix} c_+ \\ c_- \end{pmatrix}, \begin{pmatrix} c_+ \\ c_- \end{pmatrix} \right\rangle} =$$

$$= \frac{1}{2i} \frac{\bar{c}_+ c_- - \bar{c}_- c_+}{\bar{c}_+ c_+ + \bar{c}_- c_-}$$

$$\bar{S}_z = \frac{\langle \Psi, \hat{S}_z \Psi \rangle}{\langle \Psi, \Psi \rangle} = \frac{\left\langle \begin{pmatrix} c_+ \\ c_- \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} c_+ \\ c_- \end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix} c_+ \\ c_- \end{pmatrix}, \begin{pmatrix} c_+ \\ c_- \end{pmatrix} \right\rangle} =$$

$$= \frac{1}{2} \frac{\bar{c}_+ c_+ - \bar{c}_- c_-}{\bar{c}_+ c_+ + \bar{c}_- c_-}$$

S_x, S_y, S_z cannot be measured simultaneously
$[S_x, S_y] \neq 0$

$$[SU(2), \mathbb{C}P] \longrightarrow [SO(3), S^2]$$

$$\Psi \longrightarrow \langle \Psi, \hat{S}\, \Psi \rangle$$

$$g\Psi \longrightarrow \langle g\Psi, \hat{S}\, g\Psi \rangle =$$

$$= \langle \Psi, g^{-1} \hat{S}\, g \Psi \rangle$$

$$\mathrm{Ad}\, SU(2) = SO(3)$$

$$\Psi = \begin{pmatrix} c_+ \\ c_- \end{pmatrix} \sim \begin{pmatrix} u + iv \\ 1 \end{pmatrix}$$

$$\bar{S}_x = \frac{1}{2} \frac{\bar{c}_+ c_- + \bar{c}_+ c_-}{\bar{c}_+ c_+ + \bar{c}_- c_-} = \frac{1}{2} \frac{2u}{u^2 + v^2 + 1}$$

$$\bar{S}_y = \frac{1}{2i} \frac{c_+ c_- - \bar{c}_+ c_-}{\bar{c}_+ c_+ + \bar{c}_- c_-} = \frac{1}{2} \frac{2v}{u^2 + v^2 + 1}$$

$$\bar{S}_z = \frac{1}{2} \frac{\bar{c}_+ c_+ + \bar{c}_- c_-}{\bar{c}_+ c_+ + \bar{c}_- c_-} = \frac{1}{2} \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}$$

Stereographic projection

We considered physical magnitude SPIN which belongs to Quantum Mech. (it does not exist in class. mech.), and assign to this magnitude operators $\hat{S}_x, \hat{S}_y, \hat{S}_z$ in finite-dimensional unitary vector space $\mathcal{H} = \mathbb{C}^2$

What happens if we consider magnitudes such as COORDINATE?, velocity? Momentum? Energy?

These magnitudes have clear classical analogue. It turns out that operators which we will assign to them act in infinite-dimensional unitary space:

$$\mathcal{H} = C(\mathbb{R}^3) \quad \left(\mathcal{H} = L^2(\mathbb{E}^3)\right)$$

$$\langle \varphi, \psi \rangle = \int \bar{\varphi} \psi \, d^3x$$

x-coordinate — $\hat{x} : \psi \to x\psi$

y-coordinate — $\hat{y} : \psi \to y\psi$

z-coordinate — $\hat{z} : \psi \to z\psi$

p_x - momentum $\hat{p}_x : \psi \to \frac{\hbar}{i} \frac{\partial \psi}{\partial x}$

p_y - momentum $\hat{p}_y : \psi \to \frac{\hbar}{i} \frac{\partial \psi}{\partial y}$

p_z - momentum. $\hat{p}_z : \psi \to \frac{\hbar}{i} \frac{\partial \psi}{\partial z}$

$$\langle \hat{p}_x \psi, \varphi \rangle = \int \overline{\hat{p}_x \psi}\, \varphi\, d^3x =$$

$$= \int \overline{\frac{\hbar}{i}\frac{\partial \psi}{\partial x}}\, \varphi\, d^3x = -\frac{\hbar}{i}\int \overline{\frac{\partial \psi}{\partial x}}\, \varphi\, d^3x =$$

$$\overset{?}{=} \quad \ldots\ldots \quad + \frac{\hbar}{i}\int \overline{\psi}\, \frac{\partial \varphi}{\partial x}\, d^3x =$$

Here we assume that $\overline{\psi}, \varphi \to 0$ at infinity. $(*)$

$$= \int \overline{\psi} \left(\frac{\hbar}{i}\frac{\partial \varphi}{\partial x} \right) d^3x = \langle \psi, \hat{p}_x \varphi \rangle$$

We see that $\hat{p}_x$ is self-adjoint but under condition $(*)$

Another problem:

$$\hat{p}_x \psi = \lambda \psi \longrightarrow \psi = e^{ikx}$$

$$\hat{x}\psi = 2\psi \qquad \psi = \delta(x-2)$$

One can say that these functions "DO NOT EXIST" (they do not belong to the space..)

What to do???