18 X 2018

3-rd Lecture

In the previous lecture we considered average of observable

$$\overline{A} = \frac{\langle \Psi, A\Psi \rangle}{\langle \Psi, \Psi \rangle} \qquad (*)$$

If $\{\varphi_i\}$ is orthonormal basis adjusted to observable A – $\hat{A}$ – self-adjoint operator:

$$\hat{A}\varphi_i = a_i \varphi_i \qquad (a_i \text{ are real})$$

$$\langle \varphi_i, \varphi_j \rangle = \delta_{ij}$$

then

$$\overline{A} = \frac{\langle \Psi, A\Psi \rangle}{\langle \Psi, \Psi \rangle} = \frac{\sum a_i |C_i|^2}{\sum |C_i|^2} \quad (**)$$

{Here $\Psi = \sum C_i \varphi_i$ $\left(\langle \Psi, \Psi \rangle = \sum |C_i|^2\right)$

We considered $(*)$, $(**)$ for finite-dimensional case., calculating averages for spin $(\hat{S}_x, \hat{S}_y, \hat{S}_z)$

To deal with observables which have meaning in classical mechanics also we need to deal with INFINITE-DIMENSIONAL HILBERT SPACE.

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III-Lecture

We omit details related with correct definition of operators in infinite-dimensional case. Often we will use ~~analogy ifor~~ analogies for finite-dimensional case.

E.g. for infinite dimension $A : \mathcal{H}_1 \to \mathcal{H}_2$ is defined ~~not~~ on all $\mathcal{H}_1$ but on the subspace D_A which is dense in $\mathcal{H}_1$

Observables – coordinates, momenta

$$x \to \hat{X}\Psi = x\Psi \qquad p_x \to \hat{p}_x\Psi = \frac{\hbar}{i}\frac{\partial}{\partial x}\Psi$$

$$y \to \hat{y}\Psi = y\Psi \qquad p_y \to \hat{p}_y\Psi = \frac{\hbar}{i}\frac{\partial}{\partial y}\Psi$$

$$z \to \hat{z}\Psi = z\Psi \qquad p_z \to \hat{p}_z\Psi = \frac{\hbar}{i}\frac{\partial}{\partial z}\Psi$$

Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3)^*$

These operators are self-adjoint:

$$\langle \hat{x}_i\Psi, \varphi\rangle = \langle \Psi, \hat{x}\varphi\rangle \qquad \langle \hat{p}_i\Psi, \varphi\rangle = \langle \Psi, \hat{p}_i\varphi\rangle$$

$$\langle \hat{p}_i\Psi, \varphi\rangle = \int \overline{\frac{\hbar}{i}\Psi_x}\,\varphi = -\overline{\frac{\hbar}{i}}\int \overline{\Psi_x}\,\varphi = \frac{\hbar}{i}\int \overline{\Psi}\,\varphi_x = \langle \Psi, \frac{\hbar}{i}\varphi\rangle$$

* $L^2(\mathbb{R}^3)$ can be viewed as a completion of $C^2(\mathbb{R}^3) =$
$= \{$ continuous functions f : $\int \bar{f} f\, d^3x < \infty$

$$L^2(\mathbb{R}^3) = \overline{C^2(\mathbb{R}^3)}$$

($\int f_x g = -\int f g_x$ if, $f, g \in L^2(\mathbb{R})$, i.e. they are rapidly decreasing at infinity)

Try to define eigenvectors of these operators

Naive attempt:

$$\hat{x}\Psi = a\Psi, \text{ i.e. } \quad x\Psi = a\Psi \Rightarrow (x-a)\Psi = 0 \Rightarrow$$

$$\Psi = \begin{cases} 0 \text{ if } x \neq a \\ ? \text{ if } x = a. \end{cases}$$

$$\Psi = \delta(x-a)$$??? (what is it)

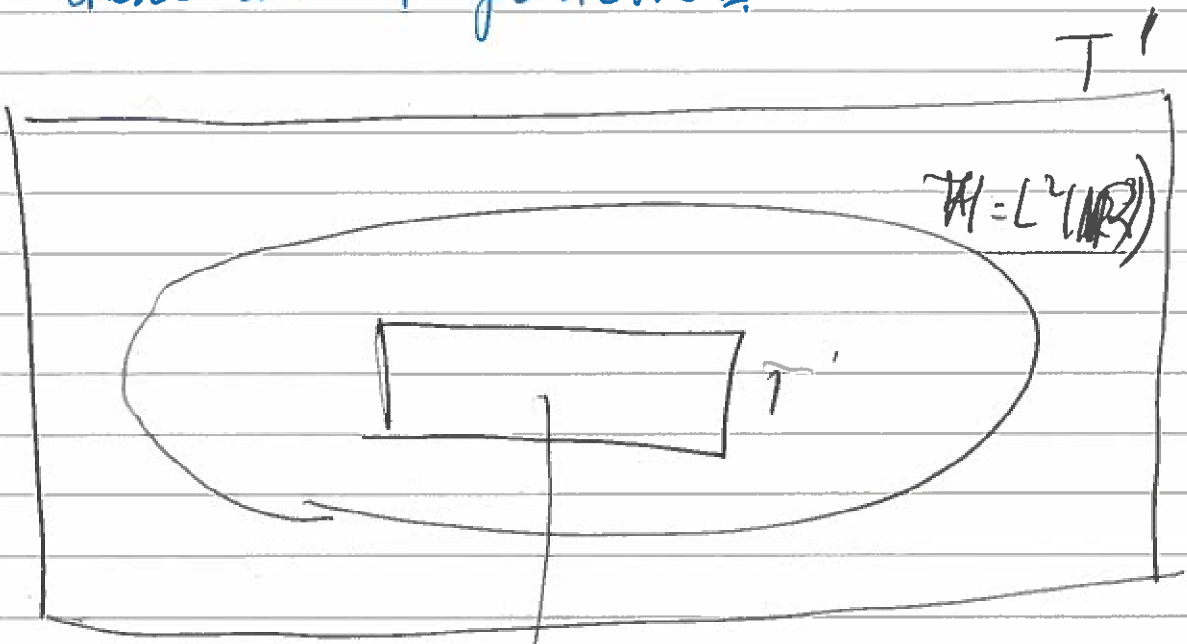
$$\hat{p}_x \Psi = p_0 \Psi \qquad \frac{\hbar}{i}\frac{\partial}{\partial x}\Psi = p_0\Psi,$$

$$\Psi = e^{\frac{i p_0 x}{\hbar}},$$

We see that eigenvector of $\hat{x}$ is a 'function' which does not exist (in a classical sence) an eigenvector of $\hat{p}_x$ is a function which is not square integrable ($f \notin \mathcal{H}$)

What to do?

Generalised functions



space of test functions

$T = \{\varphi \in C^\infty(\mathbb{R}) : \sup x^n \varphi^{(m)} < \infty\}$

T – is space of rapidly decreasing smooth functions

T' is a space of tempered distributions

T' – linear functionals on T

$T' \ni f, \quad \varphi \in T \longrightarrow f(\varphi)$

Ex. $f = \delta(x-a)$

$f(\varphi) = \int \delta(x-a)\,\varphi(x)\,dx = \varphi(a)$

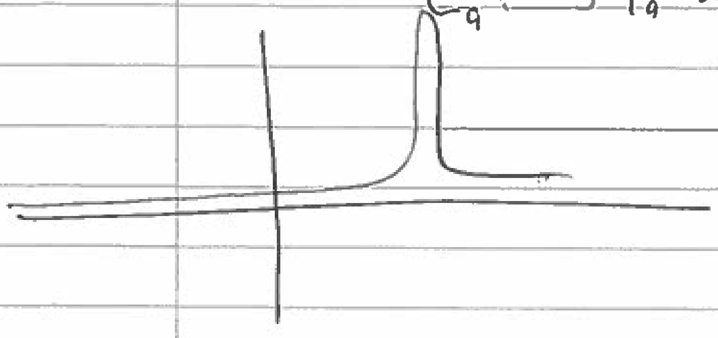
$f = \delta'(x-a)$

$f(\varphi) = \int \delta'(x-a)\,\varphi(x)\,dx = -\varphi'(a)$

Exercise: Consider

$$\varphi_a(x) = C_a e^{-\frac{(x-x_0)^2}{2a^2}}$$

C_a: $\int \varphi_a^2(x)\,dx = 1$ $\quad (C_a = \ldots\ldots)$



$$\lim_{a \to 0} \varphi_a(x) = \delta(x-a)$$

Space of generalised functions is closed ~~un~~ under Fourier transformation:

$$\psi_a(x) = C_a e^{-\frac{(x-x_0)^2}{2a^2}}$$

$$\tilde{\psi}_a(k) = \frac{C_a}{\sqrt{2\pi}} \int \psi_a(x) e^{ikx}\,dx =$$

$$= \frac{C_a}{\sqrt{2\pi}} \int e^{-\frac{x^2}{2a^2} + ikx}\,dx =$$

$$= \frac{C_a}{\sqrt{2\pi}} \int e^{-\frac{1}{2a^2}(x + ia^2k)^2 - \frac{a^2k^2}{2}}\,dx =$$

$$= \frac{C_a}{\sqrt{2\pi}} \sqrt{2}\, a \sqrt{\pi}\, e^{-\frac{a^2k^2}{2}}$$

$$\psi_a \xrightarrow{a \to 0} \delta(x) \qquad \psi(k) \to 1$$

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Generalised eigen-functions

We considered "eigenfunctions" of $\hat{X}$ and $\hat{p}_x$ which do not belong to $\mathcal{H}$. (see page 2 of this lecture)

$$X\,\delta(x-a) = a\,\delta(x-a); \quad \hat{p}_x e^{\frac{ip_0 x}{\hbar}} = p_0 e^{\frac{ip_0 x}{\hbar}}$$

Let $\mathcal{H} = L^2(\underline{M})$

$$\underset{\text{test f-ons}}{R} \subset L^2(M) \subset \underset{\text{generl. functions}}{R'}$$

f - generalised function on M $f = f(a)$ with values in $\mathcal{H}$.:

$$\forall \varphi \in R \qquad \int f(a)\,\varphi(a)\,da \in \mathcal{H}$$

We say that f is generalised eigenfunction if

$$\hat{A} f = \lambda(a)\, f(a), \text{ i.e.}^{*}$$

$$\hat{A}\left(\int f(a)\,\varphi(a)\,da\right) = \int \lambda(a)\,\varphi(a)\,da.$$

Ex. $\hat{X}\,\delta(x-a) = a\,\delta(x-a)$

$$\hat{x}\int \delta(x-a)\,\varphi(a)\,da = x\,\varphi(x)$$

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* If $\hat{A}$ is operat. self-adjoint on $\mathcal{H}$, then there exist $(M, d\mu)$:

$$\mathcal{H} \simeq L^2(M) \text{ and } \forall f(a) \in L^2(M))$$

$$\hat{A} f(a) = \lambda(a)\, f(a)$$

Exercise

$$\psi = e^{-\frac{(x-x_0)^2}{2a^2}}$$

$$\bar{x} = \frac{\langle \psi, x\psi \rangle}{\langle \psi, \psi \rangle} = \frac{\int x e^{-\frac{(x-x_0)^2}{a^2}} dx}{\int e^{-\frac{(x-x_0)^2}{a^2}}} =$$

$$= \frac{\overbrace{\int (x-x_0) e^{-\frac{(x-x_0)^2}{a^2}}}^{0} dx + x_0 \int e^{-\frac{(x-x_0)^2}{a^2}} dx}{\int e^{-\frac{(x-x_0)^2}{a^2}} dx} =$$

$$= x_0$$

Exercise

Let $\psi = \psi(x)$ be an arbitrary REAL function in H $\left(\int \psi^2 dx < \infty\right)$

$$\bar{p} = \frac{\langle \psi, \hat{p}\psi \rangle}{\langle \psi, \psi \rangle} = \frac{\frac{\hbar}{i} \int \psi \psi_x}{\int \psi^2 dx} = 0$$

$$\left(\int \psi \psi_x = - \int \psi_x \psi = 0 \right)$$

Explanation

$$e^{\frac{ipx}{\hbar}} + e^{-\frac{ipx}{\hbar}} = 2\cos\frac{px}{\hbar}$$

Momentum of real function $= 0$

It is phase which contributes to momentum.

$$\psi(x) = \rho(x) e^{i\sigma(x)}$$

$$\langle p \rangle_{average} = \frac{\langle \psi, \hat{p} \psi \rangle}{\langle \psi, \psi \rangle} =$$

$$= \frac{\int \rho(x) e^{-i\sigma(x)} \left[\frac{\hbar}{i} \rho_x e^{i\sigma(x)} - \hbar \rho \sigma_x \rho e^{i\sigma} \right] dx}{\int \rho^2(x) dx}$$

$$= \hbar^2 \int \rho^2(x) \sigma_x dx = \left\langle \frac{\partial S}{\partial x} \right\rangle_V$$

Exercise

$$\psi(x) = C e^{-\frac{(x-x_0)^2}{a^2} + \frac{i p_0 x}{\hbar}}$$

$$\bar{x} = x_0, \quad \bar{p} = p_0,$$

$$\overline{x^2} = x_0^2 + \frac{a^2}{2}, \quad \overline{p^2} = p_0^2 + \frac{\hbar^2}{2a^2}$$

$$\Delta x^2 \cdot \Delta p^2 = \frac{\hbar^2}{4}$$

One can prove that for arbitrary state

$$\Delta x^2 \cdot \Delta p^2 \geq \frac{\hbar^2}{4}$$

(Heisenberg uncertainity principle)

On the next lecture we will consider Heisenberg uncertainity principle.

To see a World in a Grain of Sand
And a Heaven in a Wild Flower
Hold Infinity in the Palm of your hand
And Eternity in an hour

William Blake