2-nd version

25 X 2018

4-th Lecture

Heisenberg uncertainty principle.

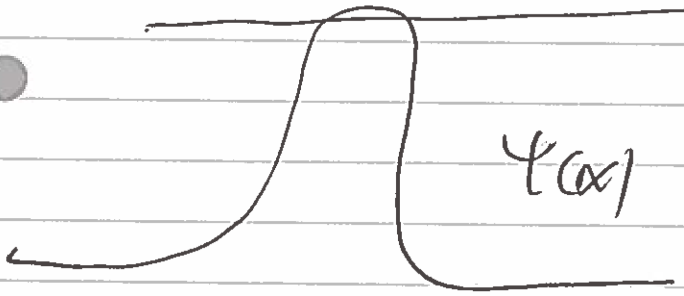
On the previous lecture we considered

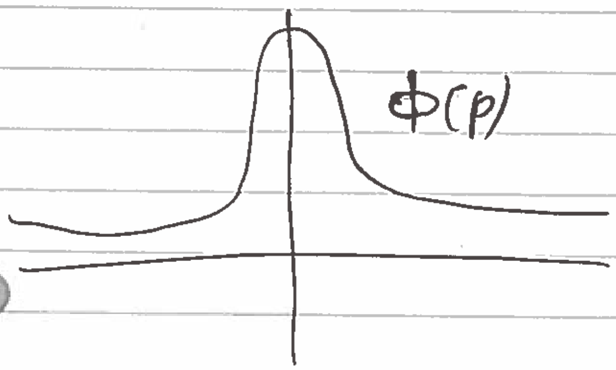
$$\Psi(x) = C e^{-\frac{(x-x_0)^2}{2a^2} + \frac{i p_0 x}{\hbar}}$$

$$\bar{x} = x_0, \quad \bar{p} = p_0$$

$$\Downarrow$$

$$\Delta x^2 \cdot \Delta p^2 = \frac{\hbar^2}{4}$$





$$\Delta x^2 \cdot \Delta p^2 \geq \frac{\hbar^2}{4}$$

This uncertainty CANNOT BE AVOIDED

2-nd version,

4-th Lecture

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Let $\hat{A}, \hat{B}$ be two observables

$$\hat{A}^+ = \hat{A}, \quad \hat{B}^+ = \hat{B}$$

1-st case $\quad \hat{A}\hat{B} = \hat{B}\hat{A}$ $\quad$ (Ex. – $\hat{A} = \hat{p}_x, \hat{B} = \hat{p}_y$; $\hat{A} = \hat{p}_x, \hat{B} = y$)

$\Downarrow$

there exists a common orthonormal basis $\{\varphi_i\}$

$$\hat{A}\varphi_i = a_i \varphi_i, \quad \hat{B}\varphi_i = b_i \varphi_i$$

In this case we say that these two observables are

simultaneously measurable

2-nd case $\quad \hat{A}\hat{B} \neq \hat{B}\hat{A}$ $\quad$ (Examples $\hat{A} = \hat{S}_x, \hat{B} = \hat{S}_y$; $\hat{A} = \hat{p}_x, \hat{B} = \hat{x}$)

$$\hat{C} = i[\hat{A}, \hat{B}]$$

Exercise: $\hat{A}, \hat{B}$ are observables $\Rightarrow$ $\hat{C}$ is observable
(See Problems 4.pdf)

Observables $\hat{A}, \hat{B}$ are not simultaneously measurable*

* still it can exist some states such that $\hat{A}, \hat{B}$ are simultaneously measurable on those states.

2n-version.

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4-th lecture

Heisenberg uncertainty principle.

Let Ψ ~~be~~ ~~defined~~ be an arbitrary state ($\langle\Psi,\Psi\rangle=1$)

Let A - be an arbitrary observable.

Dispersion of A

$$\sqrt{\Delta A_\psi^2} = \sqrt{\langle A^2\rangle_\Psi - \left(\langle A\rangle_\psi\right)^2} =$$

$$= \sqrt{\langle (A-a)^2\rangle_\Psi}\ , \quad a = \langle \hat{A}\rangle_\psi$$

Then

$$\boxed{\sqrt{\Delta A_\psi^2}\cdot\sqrt{\Delta B_\psi^2} \geq \frac{\langle C\rangle_\psi}{2}.}$$

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Proof.

$(\langle A \rangle_\psi)^2$

$$\Delta A_\psi^2 = \langle \psi, A^2 \psi \rangle - \langle \psi, A\psi \rangle^2$$

$$\Delta B_\psi^2 = \langle \psi, B^2 \psi \rangle - \langle \psi, B\psi \rangle^2$$

WLOG suppose $\langle A \rangle_\psi = \langle B \rangle_\psi = 0$

$\left(\hat{A} \to \hat{A} - a, \ \hat{B} \to \hat{B} - b, \ [\hat{A}, \hat{B}] = [\hat{A} - a, \hat{B} - b] \right.$
see Homework 1 exercises 2, 3 $\left. \right)$

$$\Delta A_\psi^2 = \langle \psi, A^2 \psi \rangle = \langle A\psi, A\psi \rangle = |A\psi|^2$$

$$\Delta B_\psi^2 = \langle \psi, B^2 \psi \rangle = \langle B\psi, B\psi \rangle = |B\psi|^2$$

$$\Delta A_\psi^2 \cdot \Delta B_\psi^2 = |A\psi|^2 |B\psi|^2 \geq$$

(Cauchy–Bunyakovsky–Schwarz inequality)

$$\underline{|\langle A\psi, B\psi \rangle|^2}$$

Now calculate $|\langle A\psi, B\psi \rangle|^2$

$$\langle A\psi, B\psi \rangle = \langle \psi, AB\psi \rangle$$

$$AB = \frac{1}{2}(AB - BA) + \frac{1}{2}(AB + BA)$$

$$\langle A\Psi, B\Psi\rangle = \langle \Psi, AB\Psi\rangle =$$

$$= \langle \Psi, \tfrac{1}{2}(AB+BA)\Psi\rangle + \langle \Psi, \tfrac{1}{2}(AB-BA)\Psi\rangle$$

$$z = a + ib$$

real number

$(AB+BA)^* = AB+BA$

imaginery number

$(AB-BA)^* = -(AB-BA)$

Hence

$$\left|\langle A\Psi, B\Psi\rangle\right|^2 \geq \left|\langle \Psi, \tfrac{1}{2}[A,B]\Psi\rangle\right|^2 =$$

$$= \left|\langle \Psi, \tfrac{1}{2}\underbrace{i[A,B]}_{C:\ C^* = C}\Psi\rangle\right|^2 = \frac{1}{4}\langle C\rangle_\Psi^2 \checkmark$$

$$\Downarrow$$

$$\Delta A_\Psi^2 \cdot \Delta B_\Psi^2 \geq \frac{1}{4}\langle C\rangle_\Psi^2$$

$\blacksquare$

~~Question, V~~

Example

$$\hat{A} = \frac{\hbar}{i}\frac{\partial}{\partial x}, \quad \hat{B} = \hat{P}_x$$

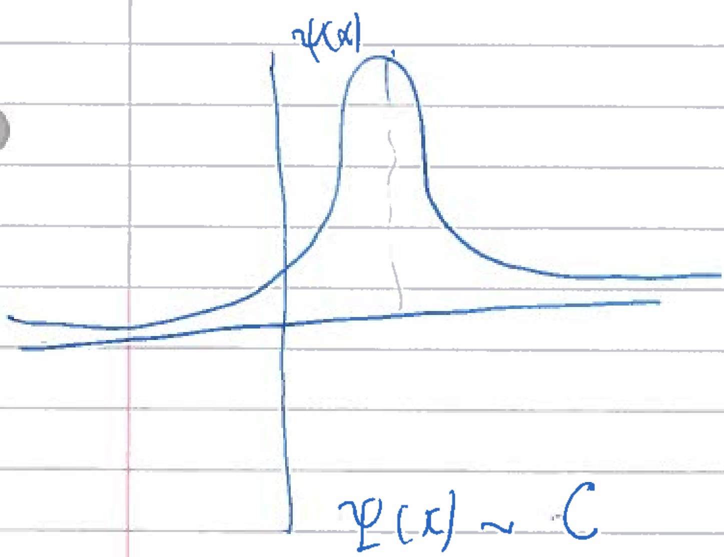
$$[\hat{A}, \hat{B}] = \frac{\hbar}{i}, \quad \hat{C} = i[\hat{A}, \hat{B}] = -\hbar$$

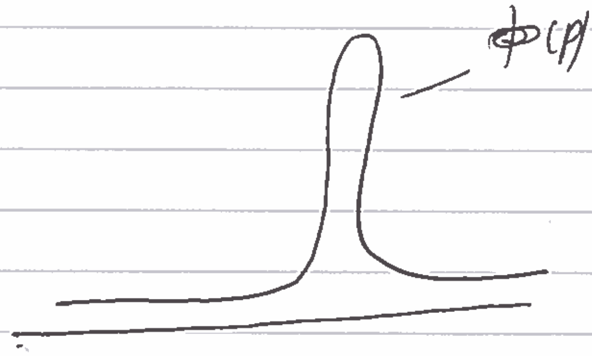
$$\Delta x^2 \cdot \Delta p^2 \geq \frac{\hbar^2}{4}$$

Question. Find a state Ψ such that $\Delta x^2 \cdot \Delta p^2$ takes minimum value..

Answer: this is gaussian.. $\psi = C e^{-\frac{(x-x_0)^2}{2a^2} + \frac{i p_0 x}{\hbar}}$

$$\Psi(x) \cancel{=} \sim \int \phi(p)\, e^{\frac{ipx}{\hbar}} d^3p$$





$\Psi(x) \sim C$ $\qquad\qquad$ $\phi(p) \sim \delta(p-p_0)$

$\Psi(x) \sim \delta(x-x_0)$ $\qquad\qquad$ $\phi(p) \sim C$

To see a World in a Grain of Sand
And a Heaven in a Wild Flower
Hold Infinity in the Palm of your hand
And Eternity in an hour

William Blake