8 November 2018
V-th lecture.

In the previous lecture we studied Heisenberg uncertainty principle.

$\hat{A}, \hat{B}$ – two observables $(\hat{A}^{\dagger} = \hat{A}, \hat{B}^{\dagger} = \hat{B})$

1-st case – they commute: $[\hat{A}, \hat{B}] = 0$

these observables could be measured SIMULTANEOUSLY, i.e. $\exists$ orth. basis $\{\varphi_i\}$:

$$\hat{A}\varphi_i = a_i \varphi_i, \quad \hat{B}\varphi_i = b_i \varphi_i$$

2-nd case $[\hat{A}, \hat{B}] \neq 0$ $\quad \hat{C} = i[\hat{A}, \hat{B}]$ $\quad (\hat{C}^{\dagger} = \hat{C})$

$$\underline{\Delta A^2 \, \Delta B^2 \geq \frac{1}{4} \overline{C}^2}$$

Example $\hat{A} = p_x$, $\hat{B} = \hat{x}$ cannot be measured simultaneously $[\hat{p}_x, \hat{x}] = \frac{\hbar}{i}$

However $\hat{A} = p_x$, $\hat{B} = \hat{y}$ can be measured simultaneously

$$\Psi = \delta(x - x_0)\, e^{\frac{i p_0 y}{\hbar}} \qquad \hat{p}_x \Psi = p_0 \Psi$$

$$\hat{x}\Psi = x_0 \Psi$$

Momentum representation

$$\Psi(x) \sim \sum C_p \phi_p$$

$$\left(\phi_p = e^{\frac{ipx}{\hbar}}\right)$$

$$\Psi(x) = \frac{1}{(2\pi\hbar)^{3/2}} \int \Phi(p)\, e^{\frac{i\vec{p}\vec{r}}{\hbar}}\, d^3p \qquad (*)$$

$\Psi(x)$ — superposition of states w

$$\Phi(p) = \frac{1}{(2\pi\hbar)^{3/2}} \int \Psi(\vec{r})\, e^{-\frac{i\vec{p}\vec{r}}{\hbar}}\, d^3r \qquad (**)$$

$$\underset{\text{coord. repr.}}{\Psi(x)} \longleftrightarrow \underset{\text{momentum repr.}}{\Phi(p)}$$

Exercise. - Check $(*)$ and $(**)$

$$\Psi(x) = \frac{1}{(2\pi\hbar)^{3/2}} \int \Phi(p)\, e^{\frac{i\vec{p}\vec{r}}{\hbar}}\, d^3p =$$

$$= \frac{1}{(2\pi\hbar)^{3/2}} \frac{1}{(2\pi\hbar)^{3/2}} \int \Psi(\vec{r}\,')\, e^{-\frac{i\vec{p}\vec{r}\,'}{\hbar}}\, d^3\vec{r}\,'\, d^3p =$$

$$= \frac{1}{(2\pi\hbar)^3} \int \Psi(\vec{r}\,')\, e^{\frac{i\vec{p}}{\hbar}(\vec{r}-\vec{r}\,')}\, d^3p\, d^3\vec{r}\,' =$$

$$= \int \Psi(\vec{r}\,')\, \delta(\vec{r}-\vec{r}\,')\, d^3\vec{r}\,' = \Psi(\vec{r}) \; *$$

$$\hat{p}_x \Psi(x) = \frac{\hbar}{i}\frac{\partial}{\partial x}\Psi(x) = \frac{\hbar}{i}\frac{\partial}{\partial x}\int \Phi(p)\, e^{\frac{i\vec{p}\vec{r}}{\hbar}}\, d^3p =$$

$$= \int p_x \Phi(p)\, e^{\frac{i\vec{p}\vec{r}}{\hbar}}\, d^3p$$

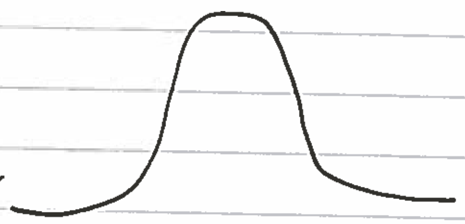
$$x\Psi(x) = \int x\,\Phi(p)\cdot e^{\frac{i\vec{p}\vec{r}}{\hbar}}\, d^3p = \int \Phi(p)\, \frac{\hbar}{i}\frac{\partial}{\partial p_x} e^{\frac{i\vec{p}\vec{r}}{\hbar}}\, d^3p =$$

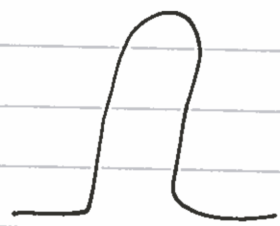
$$= -\int \frac{\hbar}{i}\frac{\partial}{\partial p_x}\Phi(p)\, e^{\frac{i\vec{p}\vec{r}}{\hbar}}\, d^3p$$

Homework.

We come to conclusion:

Coordinate represent.		Momentum represent.
$\hat{x}\Psi = x\Psi$	$\hat{x}$	$\hat{x}\Phi = i\hbar\frac{\partial}{\partial p_x}\Phi$
$\hat{y}\Psi = y\Psi$	$\hat{y}$	$\hat{y}\Phi = i\hbar\frac{\partial}{\partial p_y}\Phi$
$\hat{z}\Psi = z\Psi$	$\hat{z}$	$\hat{z}\Phi = i\hbar\frac{\partial}{\partial p_z}\Phi$
$\hat{p}_x\Psi = \frac{\hbar}{i}\frac{\partial}{\partial x}\Psi$	$\hat{p}_x$	$\hat{p}_x\Phi = p_x\Phi$
$\hat{p}_y\Psi = \frac{\hbar}{i}\frac{\partial}{\partial y}\Psi$	$\hat{p}_y$	$\hat{p}_y\Phi = p_y\Phi$
$\hat{p}_z\Psi = \frac{\hbar}{i}\frac{\partial}{\partial z}\Psi$	$\hat{p}_z$	$\hat{p}_z\Phi = p_z\Phi$

$\Psi(x) \sim$ 

$\Phi(p) \sim$ 

$|\Psi(x)|^2 \sim$ probability, that 'particle' is in a vicinity of x

$|\Phi(p)|^2 \sim$ probability, that momentum of 'particle' is in a vicinity of p

$$\Psi(x) \sim \delta(x - x_0) \qquad \Phi(p) \sim e^{\frac{ipx_0}{\hbar}}$$

$$\Psi(x) \sim e^{\frac{ip_0x}{\hbar}} \qquad \Phi(p) \sim \delta(p - p_0)$$

Schrodinger equation.

$$\vec{F} = m\vec{a}, \quad m\ddot{x}^i = -\frac{\partial U}{\partial x^i}$$

Group of invariance – $SO(3)$

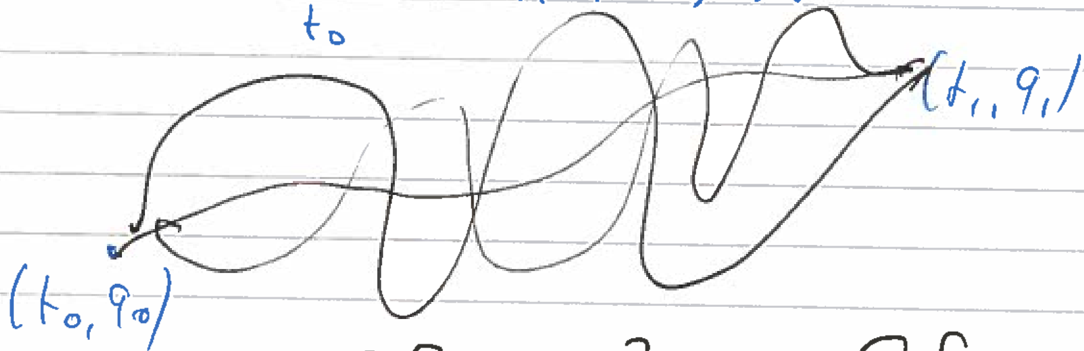
$$L(q,\dot{q}) = \frac{m\dot{q}^2}{2} - U(q) \qquad H = \frac{p^2}{2m} + U(q)$$

$$\frac{\partial L}{\partial q^i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}, \quad \dot{q}^i = \frac{\partial H}{\partial p_i}$$

Group of inv. – Diff M

Group of inveri. Canonical transform. of T^*M

$$S = \int_{t_0}^{t_1} L(q(t), \dot{q}(t))\, dt$$



$q_{class.}(t)$: $\quad S[q_{class.}] \leq S[q(t)]$

Variational principle.

S – action $\quad S = S(x_0, x, t)$:

$$\underbrace{\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial x^i}, x\right) = 0}$$

Hamilton–Jacobi equation.

$$\Downarrow$$

$\Psi(t) \in \mathcal{H}$.

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi \quad \text{(Shrodinger eq.)}$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + U(x) =$$

$$= \cancel{i\left(\frac{\hbar}{2m}\frac{\partial}{\partial x^i}\right)} = \frac{\left(\frac{\hbar}{i}\frac{\partial}{\partial x^i}\right)^2}{2m} + U(x) =$$

$$= -\frac{\hbar^2}{2m}\Delta + U(x)$$

$$i\hbar\frac{\partial\psi}{\partial t} = \left[\frac{\hat{p}^2}{2m} + U(x)\right]\psi$$

(This is in coordinate representation)

We consider three examples.

Comment

$B^+ = -B$ $\left(B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \longrightarrow e^{tB} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$

$A^+ = A \longrightarrow e^{itA} \in SU(n)$

Examples.

I - free particle $\quad H = \frac{\hat{p}^2}{2m}$

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = \frac{\hat{p}^2}{2m}\psi(x,t)$$

$$\psi(\vec{x},t) = \int \phi(\vec{p},t)\, e^{\frac{i\vec{p}\vec{r}}{\hbar}} d^3\bar{p}$$

$\psi \qquad i\hbar \frac{\partial \phi}{\partial t} = \frac{p^2}{2m}\phi$

$$\phi(p,t) = e^{-\frac{i}{\hbar}\frac{p^2}{2m}t}\left(\phi(\bar{p},t)\right)\Big|_{t=0}$$

Let $\begin{cases} \psi(\vec{r},0) = \delta(\vec{r}-\vec{r}_0) \Rightarrow \left(\phi(p) = e^{\frac{i\vec{p}\vec{r}_0}{\hbar}}\right. \\ i\hbar\frac{\partial \psi}{\partial t} = \hat{H}\psi. \end{cases}$

$$\psi(\vec{r},t) = \int e^{-\frac{i}{\hbar}\frac{p^2}{2m}t + \frac{i\vec{p}\vec{r}_0}{\hbar}} dp =$$

$$= \frac{C}{\sqrt{t}}\, e^{-\frac{(\vec{r}-\vec{r}_0)^2}{2t}}$$

action of a free particle

Quasiclassical approximation.

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi$$

$$\psi = C(x,t)\, e^{\frac{iS(x,t)}{\hbar}}$$

$$i\hbar \frac{\partial}{\partial t}\left(C e^{\frac{iS}{\hbar}}\right) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + U(x)\right) C e^{\frac{iS}{\hbar}}$$

$$\Downarrow$$

0-th order by $\hbar$

$$\underbrace{\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial x^i}\right)^2 + U(x) = 0}_{\text{Hamilton-Jacobi eq.}}$$

1-st order by $\hbar$:

$$\frac{\partial C}{\partial t} + \frac{C}{2m}\Delta S + \frac{1}{m}\nabla S\, \nabla C = 0$$

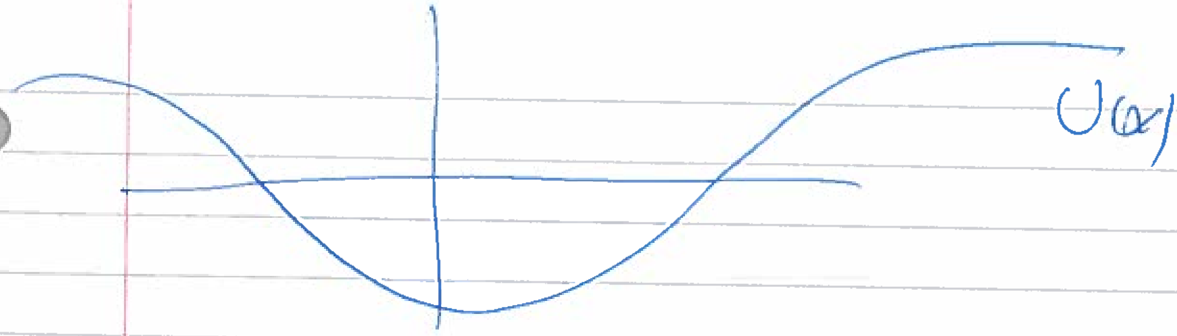
$$\Downarrow$$

$$\frac{\partial}{\partial t}|C|^2 + \frac{\partial}{\partial x^i}\left(|C|^2 \underbrace{\frac{1}{m}\frac{\partial S}{\partial x^i}}_{v^i}\right) = 0$$

$$\frac{\partial \rho}{\partial t} + \cancel{\frac{\partial}{\partial x}}\,\mathrm{div}(\rho \vec{v}) = 0$$

Continuity equation

Homework



$$i\hbar \frac{\partial \Psi}{\partial t} = \left(\frac{\hat p^2}{2m} + U(x)\right)\Psi = 0$$

$$\Psi(x,t) = \sum C_n(t)\,\varphi_n(x)$$

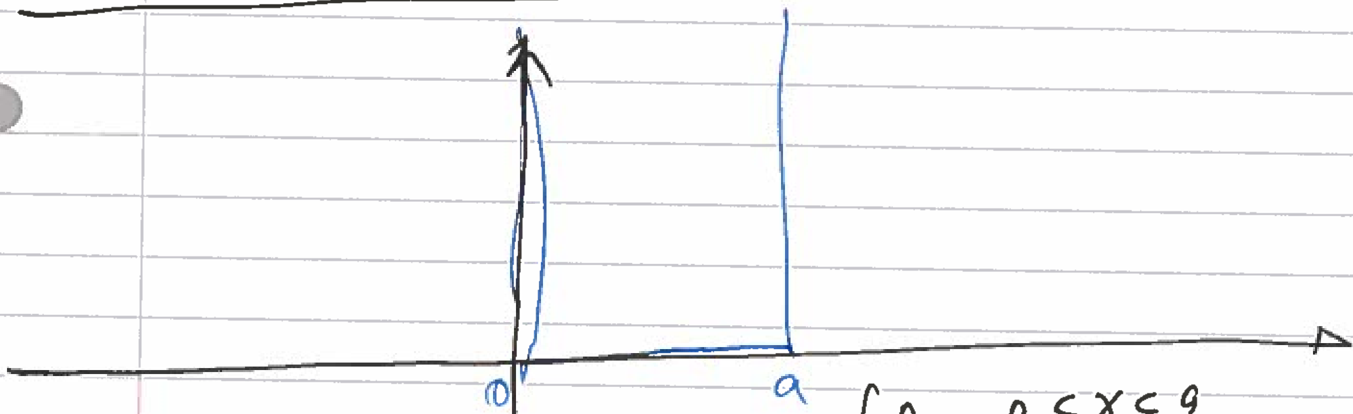
$\{\varphi_n(x)\}$: $\boxed{\hat H \varphi_n = E_n \varphi_n} \Rightarrow \boxed{i\hbar \dot C_n = E_n C_n}$

basis

$$\Downarrow$$

$$\Psi(x,t) = \sum e^{-\frac{i}{\hbar}E_n t}\, C_n(0)\,\varphi_n .$$

$\{e^{-\frac{i}{\hbar}E_n t}\varphi_n\}$ – stationary state.



$$U(x) = \begin{cases} 0, & 0 < x < a \\ \infty & x \notin (0,a) \end{cases}$$

$\{\Psi_n\}$: $\cancel{\hat p^2 \varphi''_n}$ $\quad \left(\frac{\hat p^2}{2m} + U\right)\varphi_n = E_n \varphi_n$

$$-\frac{\hbar^2}{2m}\psi_n'' = E_n \varphi_n \qquad \boxed{\varphi_n(0) = \varphi_n(a) = 0.}$$

$$\varphi_n = A\cos\frac{\sqrt{2mE_n}}{\hbar}x + B\sin\frac{\sqrt{2mE_n}}{\hbar}x$$

$A = 0$ (since $\varphi_n(0) = 0$)

$$\frac{\sqrt{2mE_n}}{\hbar} = \pi n$$

$$\boxed{E_n = \frac{\pi^2\hbar^2 n^2}{2m.}}$$

$$\psi_n = \sqrt{\frac{2}{a}}\sin\frac{\pi n x}{a.}$$

$$0 < x < a$$

$$\hat{P} : (\hat{P}\psi_n)(x) = \psi_n(a-x)$$

$$\hat{H}\hat{P} = \hat{P}\hat{H}$$

$$\Downarrow$$

~~So eigenvectors are~~ ψ_n are eigenvectors of $\hat{P}$

$$\hat{P}\varphi_n = \pm\varphi_n \qquad (\text{Homework})$$