15 November 2018

Sixth lecture

Today we consider Shrodinger equation for harmonic oscyllator:

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi,\ \hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{q}^2}{2}$$

Why harmonic oscillator???

(In quantum field theory it is the basic system, Try to give hints, why it is so, ...)

Consider arbitrary (classical mech.) Hamiltonian

$$\overset{*}{H} = \sum_{i=1}^{N}\frac{\hat{p}_i^2}{2m_i} + U(q_1, q_2, \dots, q_N)$$

H describes N degrees of freedom dyn. system

$$\begin{cases}\dfrac{dp_i}{dt} = -\dfrac{\partial U}{\partial q^i}\\[2mm] \dfrac{dq^i}{dt} = p_i\end{cases} \Longleftrightarrow \underbrace{m_i\frac{d^2q^i}{dt^2} = -\frac{\partial U}{\partial q^i}}_{\text{Newton law.}}$$

Consider potential $U = U(q_1, \dots, q_N)$ in a vicinity of stability point:

$$U = U_0 + \frac{1}{2}\sum U_{ik}(q^i - q_0^i)(q^k - q_0^k)$$

$$\left(\left.\frac{\partial U}{\partial q^i}\right|_{q_0} = 0 \ (\text{stationary point})\right)$$

$\cancel{q_i \to q_i \; q_{i0}}$

approximately:

$$U \approx U_0 + \frac{1}{2} A_{ik}(q^i - q^i_0)(q^k - q^k_0)$$

$$q^i \to q^i \to q^i_0$$

We come to

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + \frac{1}{2} \sum A_{ik} q^i q^k$$

Consider arbitrary linear transform, $q^i \to L^i_k q^k$
such that It does not change $(\dot{q}_1^2 + .. + \dot{q}_n^2)$

$$\Downarrow$$

$L^+ L = 1 \Rightarrow$ L is orthogonal matrix

$$\Downarrow$$

one can find new coordinates such that

$$H = \sum \frac{p_i^2}{2m_i} + \frac{1}{2} \sum \lambda_i (q^i)^2$$ (Linear algebra!!!)

all $\lambda_i \geq 0$ (if $\lambda_i < 0$ this is non-stable equilibrum)

denote $\lambda_i = \frac{m_i \omega_i^2}{2}$

$$\Downarrow$$

$$H = \sum \frac{p_i^2}{2m_i} + \frac{1}{2} \sum \frac{m_i \omega_i^2 q_i^2}{2} = \qquad (*)$$

$$= \sum_{i=1}^{N} \Big(\underbrace{\frac{p_i^2}{2m_i} + \frac{m_i \omega_i^2 q^{i\,2}}{2}}_{\text{free oscillator}} \Big)$$

We see that (*) describes N free (non-interacting) oscillators

Now look in detail oscillator:

$$\hat{H} = \frac{\hat{p}^2}{2} + \frac{m\omega^2 \hat{q}^2}{2} \qquad \cancel{i\hbar\partial^2}$$

in coord. present: $\Psi = \Psi(q)$, $\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial q}$, $\hat{q} = q$

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi = \left(-\frac{\hbar^2}{2m}\Psi'' + \frac{m\omega^2 q^2}{2}\Psi\right)$$

Find spectrum of this operator

$$\hat{H}\varphi_n = E_n \varphi_n \qquad \left(\Psi(q,t) = \sum_n c\, e^{\frac{i}{\hbar}E_n t}\varphi_n(q)\right)$$

Choose new coordinates ("kill dimensions")

$$q = \sqrt{\frac{\hbar}{m\omega}}\, x, \qquad \frac{d}{dq} = \frac{dx}{dq}\frac{d}{dx} = \sqrt{\frac{m\omega}{\hbar}}\frac{d}{dx}$$

$$\left(-\frac{\hbar^2}{2m}\varphi_n'' + \frac{m\omega^2 q^2}{2}\varphi_n\right) = E_n \varphi_n$$

$$\Updownarrow$$

$$\boxed{\cancel{\frac{\hbar\omega}{2}}\frac{1}{2}\left(x^2 - \frac{d^2}{dx^2}\right)\varphi_n(x) = \frac{E}{\hbar\omega}\varphi_n}$$

We look carefully on this equation

$$\hat{a} = \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right)$$

$$\hat{a}^{+} = \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right)$$

$$[\hat{a}, \hat{a}^{+}] = \frac{1}{2}\left[\left(x + \frac{d}{dx}\right), \left(x - \frac{d}{dx}\right)\right] =$$

$$\frac{1}{2}\left[\frac{d}{dx}, x\right] - \frac{1}{2}\left[x, \frac{d}{dx}\right] = 1.$$

$$\hat{a}^{+}\hat{a} + \frac{1}{2} = \frac{1}{2}\left(x - \frac{d}{dx}\right)\left(x + \frac{d}{dx}\right) + \frac{1}{2} =$$

$$= \frac{1}{2}\left(x^2 - \frac{d^2}{dx^2}\right). \qquad (*)$$

Our equation

$$\frac{1}{2}\left(x^2 - \frac{d^2}{dx^2}\right)\varphi_n = \frac{E_n}{\hbar\omega}\varphi_n \Longleftrightarrow \begin{cases} \left(a^{+}a + \frac{1}{2}\right)|\varphi\rangle = \varepsilon|\varphi\rangle \\ [a, a^{+}] = 1 \end{cases}$$

in fact $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{q} + \frac{i}{m\omega}\hat{p}\right)$

$$\hat{a}^{+} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{q} - \frac{i}{m\omega}\hat{p}\right)$$

These operators are called

creation (a^{+})

annihilation (a)

Operators

$$\hat{\Phi} = \hat{a}^+ a = \frac{m\omega}{2\hbar}\left(\hat{q} - \frac{i}{m\omega}\hat{p}\right)\left(\hat{q} + \frac{i}{m\omega}\hat{p}\right) =$$

$$= \left(\frac{m\omega^2 q^2}{2} + \frac{\hat{p}^2}{2m}\right)\hbar\omega - \frac{1}{2}$$

$$\Downarrow$$

- $$\hat{H} = \hbar\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right), \quad [\hat{a}, \hat{a}^+] = 1$$

This ~~is~~ representation is very convenient.

Find solutions:

- $$\hbar\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)|\psi_n\rangle = E_n|\psi_n\rangle$$

Consider Φ: $\hat{a}\Phi = 0$

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$\frac{1}{\sqrt{2}}\left(\frac{d}{dx} + x\right)\phi = 0$

$\phi = Ce^{-\frac{x^2}{2}}$

$$\boxed{\Phi \text{ is called Vacuum}}$$

$$\Phi = |0\rangle$$

Denote $\Phi = |0>$

(it is just notation)

$$\hat{H}|0> = (\hat{a}^+\hat{a} + \frac{1}{2})|0> = \frac{1}{2}|0>$$

since $\hat{a}|0> = 0$

Annihilation operator "kills" vacuum.

Consider

$|1> = \hat{a}^+|0>$. Use $[\hat{a}, \hat{a}^+] = 1 \Rightarrow$

$$\hat{H}|1> = (\hat{a}^+ a + \frac{1}{2})\hat{a}^+|0> =$$

$$= \left\{\hat{a}^+(\hat{a}^+ a + 1) + \frac{1}{2}a^+\right\}|0> =$$

$$= \left[\hat{a}^+\hat{a}^+ a + \frac{3}{2}\hat{a}^+\right]|0> =$$

$$= \frac{3}{2}\underbrace{\hat{a}^+|0>}_{|1>} = \frac{3}{2}|1>$$

Energy of the state $|1>$ is equal to $\boxed{\frac{3}{2}}$

Stop! – Is it true that $(|1> \neq 0)$

Check it

$$<1|1> = <\hat{a}^+|0>, \hat{a}^+|0>> =$$

$$= <0|a a^+|0> = <0|a^+ a + 1|0> =$$

$$= 1 \neq 0$$

In these calculations we just used

$$\hat{a}|0\rangle = 0.$$

annihilation operator kills the vacuum

$$\hat{a}\hat{a}^+ - \hat{a}^+\hat{a} = 1 \qquad \left([\hat{a}, \hat{a}^+] = 1\right)$$

Commutation relations

Lemma. $[\hat{H}, \hat{a}] = -\hat{a}$

$[\hat{H}, \hat{a}^+] = \hat{a}^+$

Check it!

Use it.

Let Ψ: $\hat{H}\Psi = E\Psi$

$$\hat{H}(\hat{a}^+\Psi) = (\hat{a}^+\hat{H} + \hat{a}^+)\Psi = (E+1)\hat{a}^+\Psi$$

We proved:

If Ψ has energy $E \xRightarrow{!!!} a^+\Psi$ has energy $E+1$

(It has to be checked that $a^+\Psi \neq 0$!

$$\left(\langle a^+\Psi, a^+\Psi\rangle = \underbrace{\langle \Psi, aa^+\Psi\rangle = 1 + |a\Psi|^2}_{\text{Check it!}} \geq 1\right)$$

(See exercises in the homework 6)

$$|n\rangle = \frac{(a^+)^n}{\sqrt{n!}}|0\rangle$$

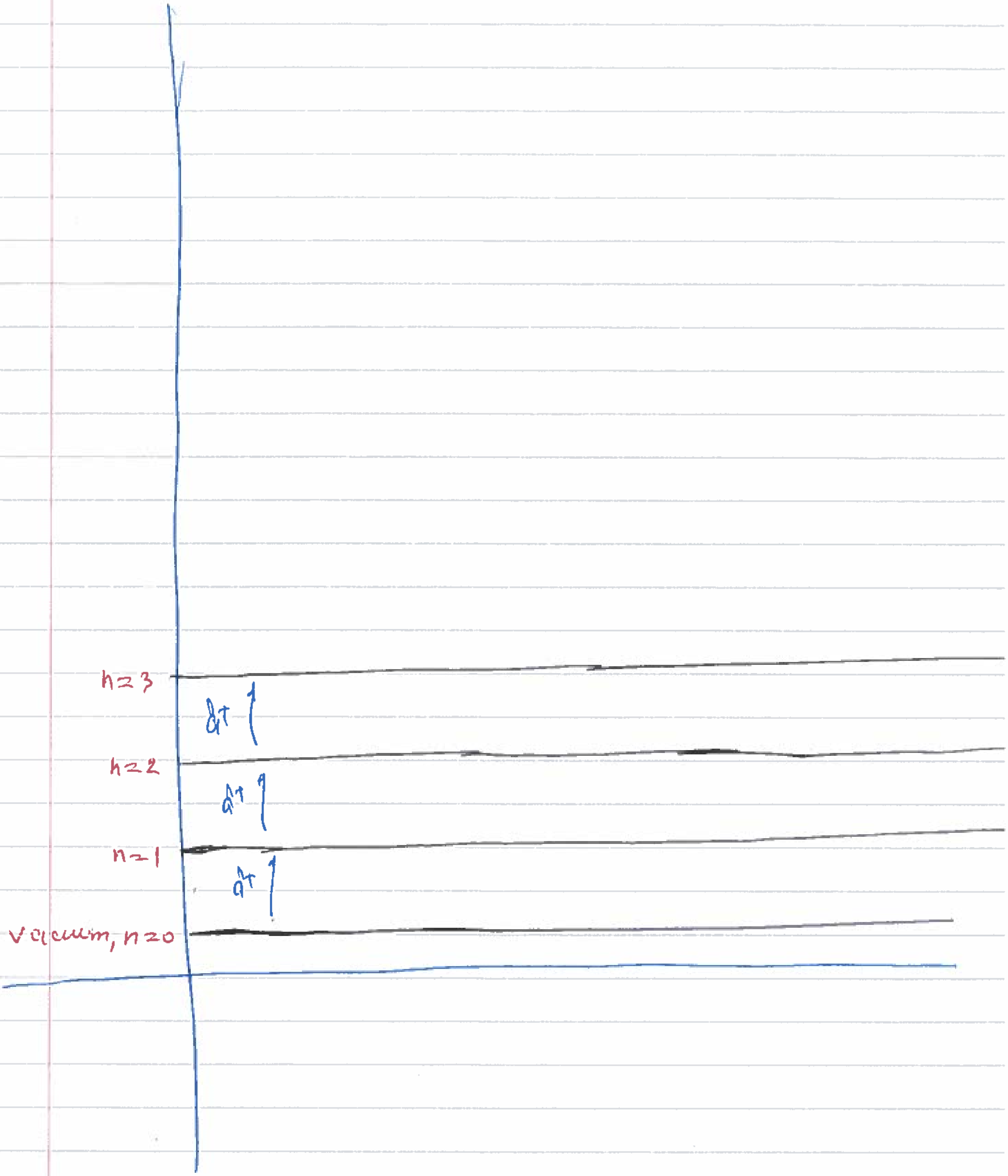
$$\langle n|m\rangle = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m, \end{cases}$$

$$\hat{a}^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$$

$\hat{a}, \hat{a}^+$ –	
– $\hat{a}^+$ transforms k-th level of oscillator to $(k+1)$-th level	$\hat{a}^+$ add particle with Energy $\hbar\omega$
$\hat{a}$: k-th level $\Rightarrow$ $(k-1)$ th level,	$\hat{a}$ 'kills' particle with energy $\hbar\omega$

We see that



Fock LADDER