22 November 2018

Seventh Lecture

In the previous lecture we explained why harmonic oscillator is so important (any system with N degrees of freedom can be approximated by N non-interacting harm. oscyll. (see the previous lecture)

Today we will explain relation between two diff. problems.

System ~~describing~~ of arbitrary number of free non-interacting particles $\Longleftrightarrow$ System of arbitrary number of free non-interacting harmonic oscillators

Let $H = \frac{p^2}{2m} + U(q)$ be Hamiltonian of particle.

$$\begin{cases} \dot{q}_i = \frac{p_i}{m} = \{H, q\} = \frac{\partial H}{\partial p_i} \\ \dot{p} = -\frac{\partial U}{\partial q_i} = \{H, q\} = -\frac{\partial H}{\partial q_i} \end{cases}$$

$\forall f = f(q, p)$

$$\frac{df}{dt} = \frac{\partial f}{\partial p_i}\dot{p}_i + \frac{\partial f}{\partial q^i}\dot{q}^i = \frac{\partial f}{\partial p_i}\left(-\frac{\partial H}{\partial q^i}\right) + \frac{\partial f}{\partial q^i}\frac{\partial H}{\partial p_i}$$

$$\dot{f} = \frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial f}{\partial p_i} = \underbrace{\{H, f\}}_{\text{Poisson bracket}}$$

Classical picture

Quantisation

$\cancel{\dot{f} = \{H, f\}}$

$\{p_i, q^i, H\}$ $\xrightarrow{\text{Quantisation}}$ $\{\hat{p}_i, \hat{q}^i, \hat{H}\}$

$\cancel{\dot{f} = \{H, f\}}$

$\{p_i, q_j\} = \delta_{ij}$ $\qquad$ $[\hat{p}_i, q_j] = \frac{\hbar}{i}$

$\dot{f} = \{H, f\}$ $\Longrightarrow$ $\boxed{i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi}$ $(*)$

Study in detail $(*)$

$$i\hbar\frac{\partial \Psi}{\partial t} = \hat{H}\Psi \qquad \left(\hat{H}\varphi_n = E_n\varphi_n\right)$$

$$\Psi(x,t) = \sum C_n(t)\,\varphi_n(x)$$

$$i\hbar\frac{\partial \Psi}{\partial t} = i\hbar\sum \dot{C}_n(t)\varphi_n(x) = \hat{H}\left(\sum C_n(t)\varphi_n(x)\right)$$

$\hat{H}\varphi_n = E_n\varphi_n$

$\Downarrow$

$$i\hbar\frac{dC_n}{dt} = E_n C_n$$

Diff. equations on $\{C_n\}$

$C_n - \{n = 1, 2, 3, \ldots\}$ new coordinates.

$$i\hbar \frac{dc_n}{dt} = E_n c_n \qquad (*)$$
$$i\hbar \frac{dc_n^*}{dt} = E_n c_n^*$$

c_n, c_n^* — complex coordinates, $\mathcal{H}$

Can we find Hamiltonian (classical!) and Poisson brackets $\{ , \}$ such that

$$\frac{dc_n}{dt} = \{\mathcal{H}, c_n\} \qquad (**)$$
$$\frac{dc_n^*}{dt} = \{\mathcal{H}, c_n^*\}$$

and these systems be

equivalent?

Consider a space (symplectic phase space) $\{c_n\}$ with complex coordinates

$$(c_1, c_1^*, c_2, c_2^*, \ldots, c_n, c_n^*, \ldots)$$

and with Poisson bracket

$$\boxed{\{c_n, c_n^*\} = \frac{i}{\hbar}} \qquad \mathcal{H} = \sum E_n c_n^* c_n$$

$$\dot{c}_n = \frac{1}{i\hbar} E_n c_n = \{\mathcal{H}, c_n\} = \{\sum E_k c_k^* c_k, c_n\}$$

$$= E_k c_k \underbrace{\{c_k^*, c_n\}}_{\delta_{kn}} + E_k c_k^* \underbrace{\{c_k, c_n\}}_{0}$$

We come to

$$\{c_n, c_k^*\} = \frac{i}{\hbar}, \quad \{c_n, c_m\} = \{c_n^*, c_m^*\} = 0$$

$$\{f, g\} = \frac{i}{\hbar}\left(\frac{\partial f}{\partial c_n}\frac{\partial g}{\partial c_n^*} - \frac{\partial f}{\partial c_n^*}\frac{\partial g}{\partial c_n}\right)$$

$$\dot{f} = \{\mathcal{H}, f\} = \frac{i}{\hbar}\left(\frac{\partial \mathcal{H}}{\partial c_n}\frac{\partial f}{\partial c_n^*} - \frac{\partial \mathcal{H}}{\partial c_n^*}\frac{\partial f}{\partial c_n}\right)$$

These are just classical equations (but the system has infinite number of degrees of freedom!!!)

$$i\hbar \dot{c}_n = E_n c_n \qquad \sim \qquad \dot{c}_n = \{\mathcal{H}, c_n\}$$

$$i\hbar \dot{c}_n^* = E_n c_n^* \qquad \qquad \dot{c}_n^* = \{\mathcal{H}, c_n^*\}$$

Check it

$$\{\mathcal{H}, c_n\} = \{\sum_k E_k c_k^* c_k, c_n\} =$$

$$= E_k c_k \{c_k^*, c_n\} = E_k c_k \left(-\frac{i}{\hbar}\delta_{kn}\right)$$

$$\boxed{\dot{c}_n = -\frac{i}{\hbar} E_n c_n}$$

So we see that $(\mathcal{H} = \sum E_n c_n^* c_n, \{,\})$ is Mechanical system which describes Quantum particle

Quantum particle

$$\hat{H} = \frac{\hat{p}^2}{2m} + U(q)$$

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi$$

= Mechanical system with infinite number degrees of freedom.

Next step

Quantise this Mech. system:

Free particle

$\hat{H} = \frac{\hat{p}^2}{2m} + U(q)$

$i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi$

$=$

Mec. system

$\mathcal{H} = \sum_{n=1}^{\infty} E_n C_n^* C_n$

$\dot{C}_n = \{\mathcal{H}, C_n\}$

$\{C_n, C_m^*\} = \frac{i}{\hbar}$

$\Rightarrow$

Quantisation

??

$$\hat{\mathcal{H}} = \sum E_n \hat{C}_n^* C_n$$

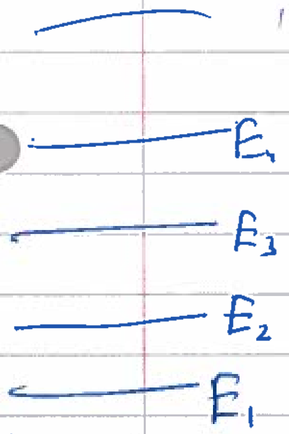
$$[C_n, C_m^*] = \mathbb{1} = \frac{i}{\hbar} \cdot \frac{\hbar}{i}$$

??? We come to something related with harm. oscillator???

$$\hat{\mathcal{H}} = \sum_{n=1}^{N} E_n \hat{c}_n^* c_n$$

$(N \to \infty)$

System describes N non interacting particles



$\hat{H}\varphi_k = E_k \varphi_k$

there are n_1 particles on the level E_1

There are n_k particles, such that every particle has energy E_k

It is added the particle on the ~~level~~ second ~~level~~ level.

$$\hat{\mathcal{H}} = \sum_{n=1}^{N} E_n \left(\hat{a}_n^+ a_n + \frac{1}{2}\right)$$

$(N \to \infty)$

System describes N non-interacting harm. oscillators.

Ψ

$|n_1 n_2 \dots n_N\rangle$

First oscyll. has energy $\hbar\omega_n\left(n + \frac{1}{2}\right)$

$$\boxed{E_n = \hbar\omega_n}$$

k-th oscillator has energy $\hbar\omega_k\left(n_k + \frac{1}{2}\right)$

the k-th oscyllator has energy E_k

$a_2^+|n_1 n_2 \dots k n_k\rangle = \sqrt{n_2+1}\,|n_1 n_2 \dots n_{11}\rangle$

the energy of second oscillator is increasing on $\hbar\omega_2$

$$\hat{H} = \frac{\hat{p}^2}{2m} + U(q), \quad \hat{H}\varphi_n = E_n \varphi_n$$

$(E_n = \hbar\omega_n)$

Two interpretations

$$\Psi = |0\ 1\ 3\ 2\ 0 \dots\rangle$$

no particle with energy E_1

first oscillator in the vacuum state
$E = \hbar\omega_1(0 + 1/2)$

one particle with energy E_2

second oscillator in the first state
$E = \hbar\omega_2(1 + 1/2)$

three particles with energy E_3

third oscillator in the third state
$E = \hbar\omega_3(3 + \frac{1}{2})$

two particles with energy E_4

fourth oscillator in the second state
$E = \hbar\omega_4(2 + 1/2)$

No particles with other energies

all other oscillators are in the vacuum state.

$$\Psi = \varphi_2(x_1) \times \varphi_3(y_1)\varphi_3(y_2)\varphi_3(y_3) \times \varphi_4(z_1)\varphi_4(z_2) + \text{symmetrisation}$$

$$\Psi = e^{-\frac{x_1^2}{2}} \cdot H_1(x_2) e^{-\frac{x_2^2}{2}} \cdot H_3(x_3) e^{-\frac{x_3^2}{2}} \cdot H_2(x_4) e^{-\frac{x_4^2}{2}} \cdot \prod_{k=5}^{\infty} e^{-\frac{x_k^2}{2}}$$

(9) App: What is it PHONON ???

Consider lattice of particles which have small oscillations around equilibrium points:

Sure these particles <u>are not free</u>

however one can find new coordinates (collective coordinates)

$(p_i, q_i) \longrightarrow (P_i, Q_i)$ such that in these coordinates

$$\ddot{Q}_i + \omega_i^2 Q_i = 0.$$

(harmonic oscillator)

Why? — It is linear algebra

$$H = \sum \frac{p_i^2}{2m} + U(q_i) = \sum \frac{p_i^2}{2m} + \underbrace{A_{km} q^k q^m}_{\text{quadratic form}}$$

Non-interacting particles $=$ Diagonal form

$$\Psi = |n_1\, n_2\, n_3 \ldots\ldots \rangle$$

there are n_i "quasiparticles" with energy $E_i = \hbar\omega_i$

i-th ~~os~~ free oscillator is the state n_i

$$E = \hbar\, \omega_i (n_i + \tfrac{1}{2})$$