29 November 2018

VIII – th lecture.

Recall represent. theory

$[G, V]$ $\quad G \xrightarrow{T} GL(V)$

linear representation of group G in V *

Let $[G, V_1]$, $[G, V_2]$

$$\begin{tikzcd} V_1 \arrow[r, "\sigma"] \arrow[d, dashed, "g"] & V_2 \arrow[d, dashed, "g"] \\ V_1 \arrow[r, "\sigma"] & V_2 \end{tikzcd}$$

~~σ/2~~ – morphism : σ – morphism

$$\sigma \circ T_1(g) = T_2(g) \circ \sigma$$

Theorem If $[G, V_1]$, $[G, V_2]$ are irreducible representations, then σ is isomorphism (if it is not trivial)

Proof: $\operatorname{Im} \sigma \leq V_2$, $\ker \sigma \leq V_1 \Rightarrow$ $\operatorname{Im} \sigma = 0$, or V_2

$\ker \sigma = 0$, or V_1

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* In fact one have to consider projective representations. However for groups that we consider it suffices to consider linear represent. (Bargmann Theorem).

Every space can be decomposed on irreducible invariant subspaces - bricks.

Thesis: Elementary particle is irreducible representation of Poincare group.

Wigner

Schur lemma.

Let $[G, V]$ be irreducible representation of G in complex vector space V, then $\sigma: V \to V$ is morphism $\Longrightarrow$ σ - scaler operator

$$\sigma = \lambda I.$$

Proof $\exists \vec{x}_0 \neq 0: \quad \sigma(\vec{x}_0) = \lambda \vec{x}_0, \quad \vec{x}_0 \in \ker(\sigma - \lambda I)$

$[G, V]$ irreducible $\Rightarrow$ $\ker(\sigma - \lambda I) = V$, or $\emptyset$,
however $\vec{x}_0 \in \ker(\sigma - \lambda I) \Rightarrow \ker(\sigma - \lambda I) = V \Rightarrow$

$$\sigma \equiv \lambda I \quad \blacksquare$$

Corollary:
For Complex irreducible representation of abelian group has to be ONE-DIMENSIONAL

$\Downarrow$ Consequence in Quantum Mechanics

Irreducible representation of translation group ~~are~~ is one-dimensional. $\Longrightarrow$

Proof: $\forall g_1, g_2 : \quad T_{g_1} \circ T_{g_2} = T_{g_2} \circ T_{g_1}$

~~fix $g_1 = g_0$~~

$$\Downarrow$$

$$\sigma : \vec{X} \Longrightarrow T_{g_1} \vec{X} \text{ is morphism}$$

$$\Downarrow$$

$$T_{g_1} \vec{X} = \chi(g_1) \vec{X}$$

$$\Downarrow$$

Invariant space is spanned by $\vec{X}$ ■

Example

$$(T_{\vec{a}} f) = f(\vec{x} + \vec{a})$$

Representation of translations group in the space of functions.

eigenfunctions: $\{ e^{i\vec{k}\vec{x}} \}$:

$$T_{\vec{a}}\, e^{i\vec{k}\vec{r}} = e^{i\vec{a}\vec{r}} \cdot e^{i\vec{k}\vec{r}}$$

Irreducible representations of translation group are one-dimensional

$$\Downarrow$$

Fourier expansion is the expansion of the function over irreducible representations

Now consider

$$[SO(3), C(\mathbb{R}^3)]$$

not-abelian group!

$K[x, y, z]$ - polynomials on $\mathbb{R}^3$

$$A_m = \{P : P(\lambda \vec{r}) = \lambda^m P(\vec{r})\}$$

A_m — subspace of polynomials of order m,

A_m is invariant subspace of $SO(3)$ action.

$$P = \sum P_{i_1 \ldots i_m} x^{i_1} \ldots x^{i_m}$$

$$\downarrow$$

$$P^g = \sum P_{i_1 \ldots i_m} T^{i_1}_{j_1} T^{i_2}_{j_2} \ldots T^{i_m}_{j_m} x^{j_1'} \ldots x^{j_m'}$$

$T: \quad x^{i_*} = T^i_j x^j, \quad T \in SO(3)$

orthogonal transformation

A_m is not invariant subspace $(m>1)$

$$P(\vec{r}) = x^2 + y^2 + z^2$$

Consider

$$H_m = \{ P : \ P \in A_m, \ \Delta P = 0 \}$$

H_m – harmonic polynomials of weight m.

$$A_2 = H_2 \oplus r^2 H_0$$

$$P_{ik} = \left(P_{ik} - \frac{1}{3}\delta_{ik} P_{rr}\right) + \frac{1}{3}\delta_{ik} P_{rr}$$

$$H_m \ni P = \sum P_{i_1 i_2 \ldots i_m} x^{i_1} \ldots x^{i_m}$$

$\{P_{i_1, \ldots i_m}\}$ – symmetric ~~tensor~~ traceless tensor

$$P_{rr i_3 \ldots i_m} = 0.$$

Consider in A_m
scalar product:

$$\langle x^k y^m z^n, x^{k'} y^{m'} z^{n'} \rangle =$$

$$= k!\,m!\,n!\,\delta_{kk'}\,\delta_{mm'}\,\delta_{nn'}$$

$$\left(\frac{\partial}{\partial x}\right)^* = x, \quad \left(\frac{\partial}{\partial y}\right)^* = y, \quad \left(\frac{\partial}{\partial z}\right)^* = z$$

$$\Delta^* = x^2 + y^2 + z^2$$

$$\Downarrow$$

Theorem $\Delta : A_m \to A_{m-2}$
$H_m = \ker \Delta|_{A_m}$

$$A_m = \cancel{A_{m-2}}\; H_m \oplus r^2 A_{m-2}$$

$$\underbrace{A_m = H_m \oplus r^2 H_{m-2} \oplus r^4 H_{m-4} \oplus \dots}_{\text{expansion over spherical harmonic}}$$

$$\left(H_m \ni P\big|_{S^2} - \text{spheric. harmonic} \right)$$

$$\phi(\vec{r}) = \phi_0(r) + \phi_i(r)x^i + \phi_{ik}(r)x^i x^k + \phi_{ikm}(r)x^i x^k x^m + \dots$$

$$\phi_{ii} = 0, \quad \phi_{iim} = 0, \quad \dots$$

<u>Th.</u> All $\{H_m\}$ are irreducible subspaces.

~~They posses exactly~~

A_m possesses $\left[\frac{m}{2}\right] + 1$ invariant sub spaces.

On the other hand it possesses $\left[\frac{m}{2}\right]+1$ SO(2) invariant polynomials

$$\{z^m, z^{m-2} w\bar{w}, z^{m-4}(w\bar{w})^2, \dots\}$$

$w = x + iy$

Hence all H_m are irreducible!

$$\dim H_m = \dim A_m - \dim A_{m-2} =$$
$$= C^2_{m+2} - C^2_{m-2} = \underline{2m+1}$$

$\{H_m\}$ - space of polynomials:

$\hat{L}$ $\qquad \hat{L}^2 P = m(m+1) P, \quad P \in H_m,$

$$\hat{L}^2 = -r^2 \Delta + \hat{E}^2 + \hat{E}$$

Legendre polynomials.

$$\hat{L}_x = i\left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}\right), \quad \hat{L}_y = i\left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z}\right), \ldots$$

$$[\hat{L}_x, \hat{L}_y] = -i \hat{L}_z$$