

29 November 2018

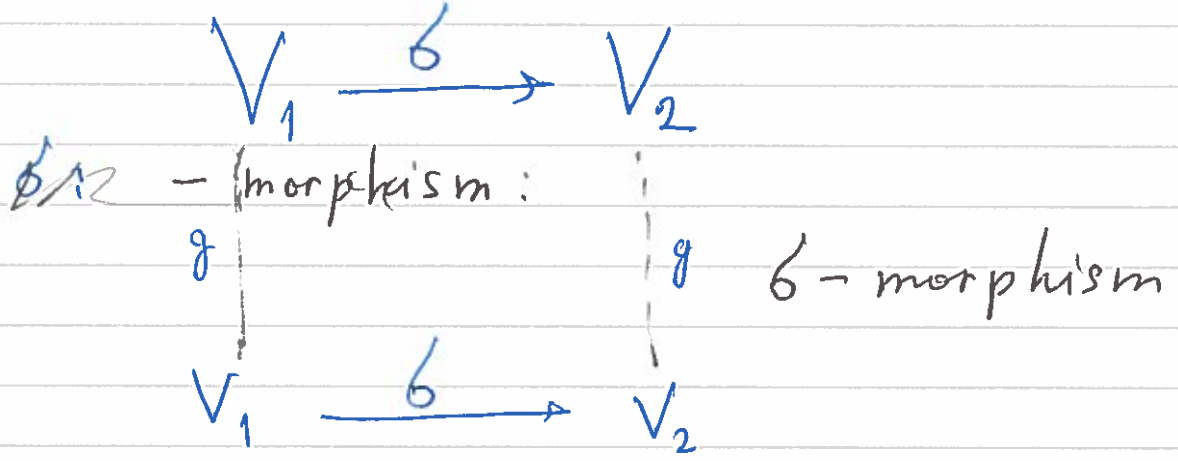
VIII - th lecture.

Recall represent. theory

$$[G, V] \quad G \longrightarrow GL(V)$$

linear representation of group G in V^n

Let $[G, V_1], [G, V_2]$



$$\sigma \circ T_1(\vartheta) = T_2(\vartheta) \circ \sigma$$

Theorem If $[G, V_1], [G, V_2]$ are irreducible representations, then σ is isomorphism

(if it is not trivial)

Proof: $\text{Im } \sigma \leq V_2, \text{ker } \sigma \leq V_1 \Rightarrow$

$$\begin{array}{l}
 \text{Im } \sigma = 0, \overset{\circlearrowright}{V_2} \\
 \text{ker } \sigma = 0, \overset{\circlearrowleft}{V_1}
 \end{array}$$

* In fact one have to consider projective representations. However for groups that we consider it suffices to consider linear represent. (Bargmann's Theorem).

Every space can be decomposed on irreducible invariant subspaces - bricks.

Thesis: Elementary particle is irreducible representation of Poincare group. Wigner

Schur lemma.

Let $[G, V]$ be irreducible representation of G in complex vector space V , then

$\sigma: V \rightarrow V$ is morphism $\implies \sigma$ - scalar operator
 $\sigma = \lambda I$.

Proof

$$\exists \vec{x}_0 \neq 0: \sigma(\vec{x}_0) = \lambda \vec{x}_0, \vec{x}_0 \in \ker(\sigma - \lambda I)$$

$[G, V]$ irreducible $\implies \ker(\sigma - \lambda I) = V$, or \emptyset ,
however $\vec{x}_0 \in \ker(\sigma - \lambda I) \implies \ker(\sigma - \lambda I) = V \implies$

$$\sigma \equiv \lambda I \quad \square$$

Corollary:

For complex irreducible representation
of abelian group

has to be ONE-DIMENSIONAL

Consequence
in Quantum Mechanics

Irreducible representation of translation
group is one-dimensional. \Rightarrow

→ Proof: $\forall g_1, g_2: T_{g_1} \circ T_{g_2} = T_{g_2} \circ T_{g_1}$

fix $g_1 = g_0$

\Downarrow
 $\sigma: \vec{X} \Rightarrow T_{g_1} \vec{X}$ is morphism

\Downarrow
 $T_{g_1} \vec{X} = X_{g_1} \vec{X}$

Invariant space is
spanned by \vec{X} \square

Example

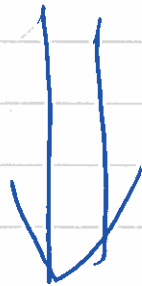
$$(T_{\vec{a}} f) = f(\vec{x} + \vec{a})$$

Representation of translations group
in the space of functions,

eigenfunctions: $\{e^{i\vec{k}\cdot\vec{r}}\}$:

$$T_{\vec{a}} e^{i\vec{k}\cdot\vec{r}} = e^{i\vec{a}\cdot\vec{k}} \cdot e^{i\vec{k}\cdot\vec{r}}$$

Irreducible representations of translation
group are one-dimensional



Fourier expansion
is the expansion of the function
over irreducible representations

Now consider

$$[SO(3), C(\mathbb{R}^3)]$$

not-abelian group!

$K[x, y, z]$ - polynomials on \mathbb{R}^3

$$A_m = \{P: P(\lambda \vec{r}) = \lambda^m P(\vec{r})\}$$

A_m - subspace of polynomials of order m ,

A_m is invariant subspace of $SO(3)$ action.

$$P = \sum_{i_1, \dots, i_m} P_{i_1, \dots, i_m} x^{i_1} \dots x^{i_m}$$



$$P^g = \sum_{j_1, \dots, j_m} P_{i_1, \dots, i_m} T_{j_1}^{i_1} T_{j_2}^{i_2} \dots T_{j_m}^{i_m} x^{j_1} \dots x^{j_m}$$

$$T: x^{i_j} = T_{j'}^{i_j} x^{j'}, \quad T \in SO(3)$$

orthogonal transformation

A_m is not invariant subspace ($m > 1$)

$$P(\vec{r}) = x^2 + y^2 + z^2$$

Consider

$$H_m = \{ P : P \in A_m, \Delta P = 0 \}$$

H_m - harmonic polynomials
of weight m .

$$A_2 = H_2 \oplus r^2 H_0$$

$$P_{ik} = \left(P_{ik} - \frac{1}{3} \delta_{ik} P_{rr} \right) + \frac{1}{3} \delta_{ik} P_{rr}$$

$$H_m \ni P = \sum P_{i_1 i_2 \dots i_m} x^{i_1} \dots x^{i_m}$$

$\{ P_{i_1 \dots i_m} \}$ - symmetric ~~tensor~~

traceless tensor

$$P_{rr i_3 \dots i_m} = 0.$$

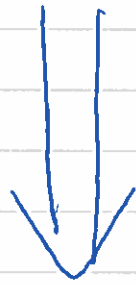
Consider in A_m
scalar product:

$$\langle x^k y^m z^n, x^{k'} y^{m'} z^{n'} \rangle =$$

$$= k! m! n! \delta_{kk'} \delta_{mm'} \delta_{nn'}$$

$$\left(\frac{\partial}{\partial x}\right)^k = X, \quad \left(\frac{\partial}{\partial y}\right)^m = Y, \quad \left(\frac{\partial}{\partial z}\right)^n = Z$$

$$\Delta^* = x^2 + y^2 + z^2$$



Theorem $\Delta: A_m \rightarrow A_{m-2}$
 $H_m = \ker \Delta|_{A_m}$

$$A_m = \text{ker } \Delta \oplus \Gamma^2 A_{m-2}$$

$$A_m = H_m \oplus r^2 H_{m-2} \oplus r^4 H_{m-4} \oplus \dots$$

expansion over spherical harmonics

$$(H_m \ni P|_{S^2} - \text{spheric. harmonic})$$

$$\begin{aligned} \phi(\vec{r}) &= \phi_0(r) + \phi_2(r)x^d + \phi_{ik}(r)x^i x^k + \\ &+ \phi_{ikm}(r)x^i x^k x^m + \dots \\ \phi_{00} &= 0, \quad \phi_{i0m} = 0, \dots \end{aligned}$$

Th. All $\{H_m\}$ are irreducible subspaces,

They possess exactly

A_m possesses $\left[\frac{m}{2}\right] + 1$ invariant subspaces,

On the other hand it possesses $\left[\frac{m}{2}\right] + 1$ $SO(2)$ invariant polynomials

$$\{z^m, z^{m-2} w \bar{w}, z^{m-4} (w \bar{w})^2, \dots\}$$

$w = x + iy$
Hence all H_m are irreducible!

$$\dim H_m = \dim A_m - \dim A_{m-2} =$$

$$= C_{m+2}^2 - C_{m-2}^2 = \underline{2m+1}$$

$\{H_m\}$ - space of polynomials:

$$\hat{L}^2 P = m(m+1)P, \quad \varphi \in H_m,$$

$$\hat{L}^2 = -r^2 \Delta + \hat{E}^2 + \hat{E}$$

Legendre polynomials.

$$\hat{L}_x = i \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right), \quad \hat{L}_y = i \left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right),$$

$$[\hat{L}_x, \hat{L}_y] = -i \hat{L}_z$$